Introdução à Mecânica dos Materiais Compósitos Estruturais

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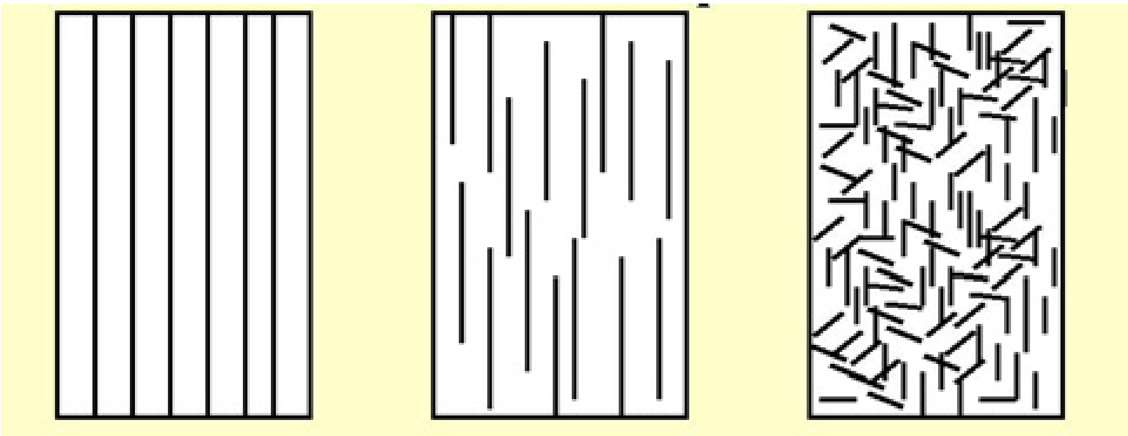
ENG 1704 – Mecânica dos Sólidos II

I. Materiais Compósitos

Referência: Daniel, I. M., and Ishai, O., *Engineering Mechanics of Composite Materials*, 2nd ed., Oxford University Press, 2005

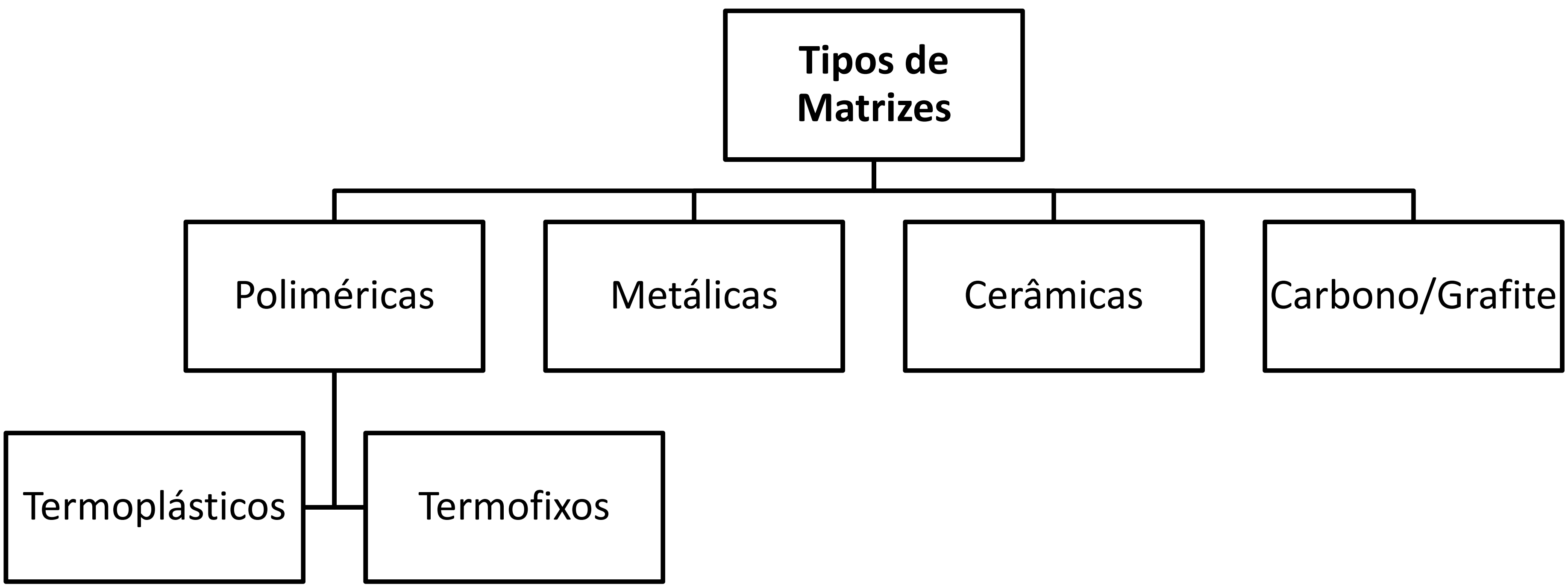
Materiais Compósitos

- Materiais fabricados a partir de dois ou mais constituintes
- Nosso interesse:
  – Compósitos estruturais reforçados por fibra (Matriz + Fibras)

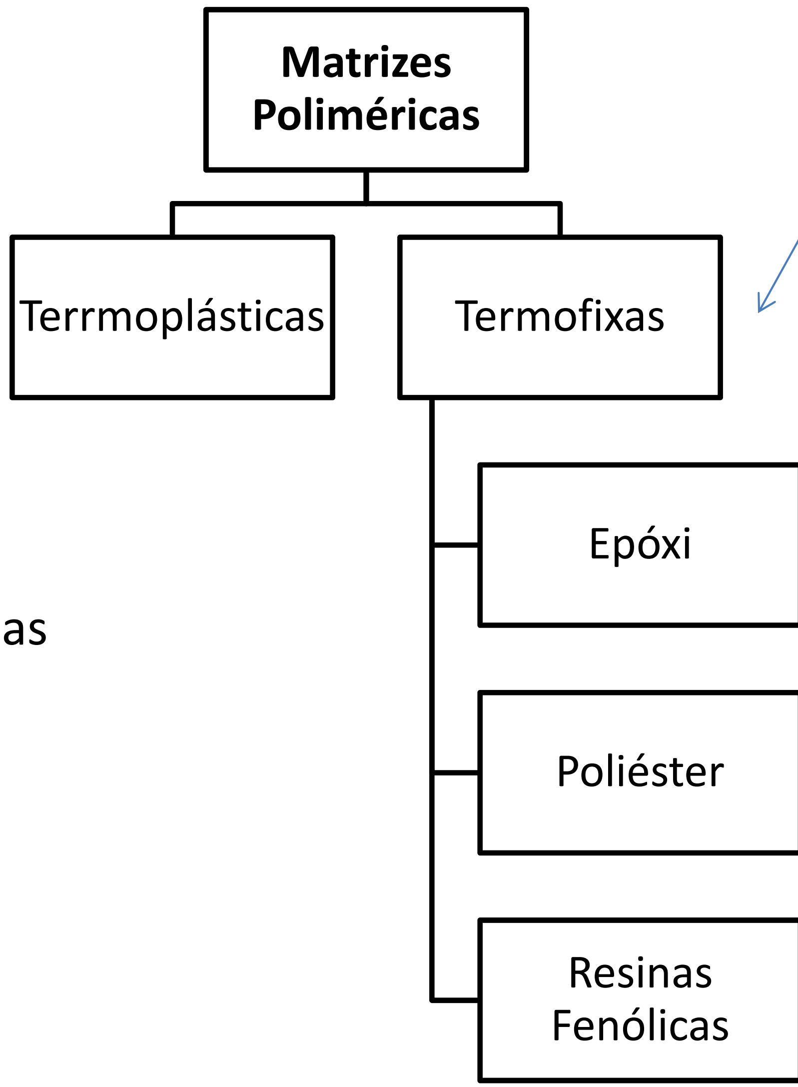


Materiais Compósitos

Matriz: *agregar o reforço* e *distribuir tensões*



Materiais Compósitos



Mais utilizadas em compósitos estruturais

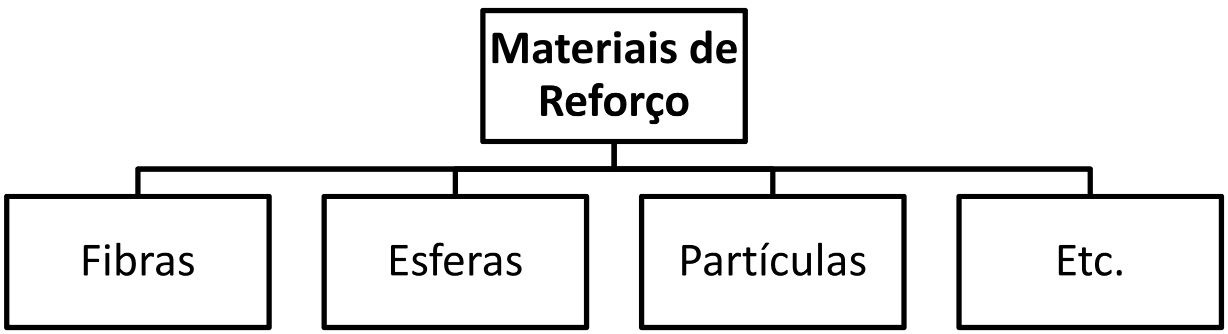
Termoplásticos

- Podem ser conformados várias vezes quando aquecidos
- Recicláveis

Termofixos

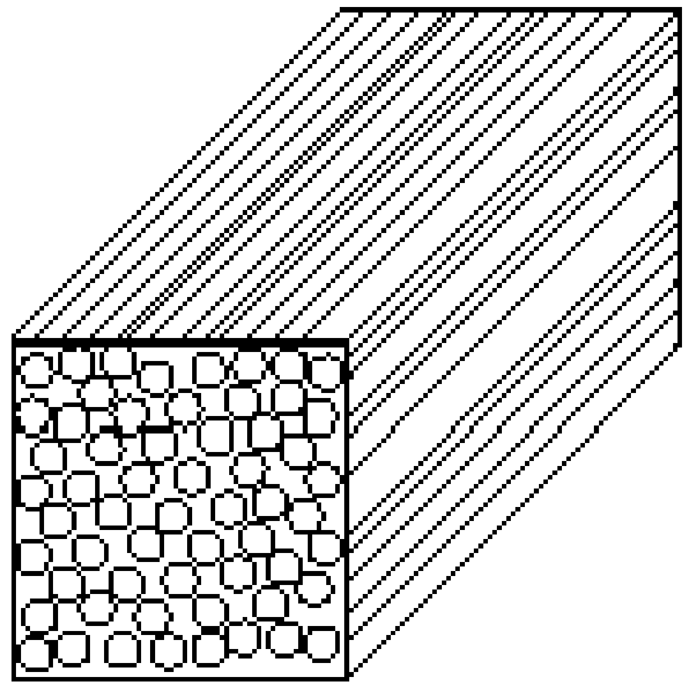
- Resistência a temperatura
- Insolúveis e infusíveis
- Estrutura tridimensional com ligações crudzadas
- Mais rígidos

Materiais Compósitos



Materiais Compósitos

- Fibras
  - Vidro
  - Carbono
  - Metais
  - Alumina
  - Boro
  - Aramida
  - Carbureto de Silício
  - Quartzo/Sílica
  - Grafite
  - Etc.

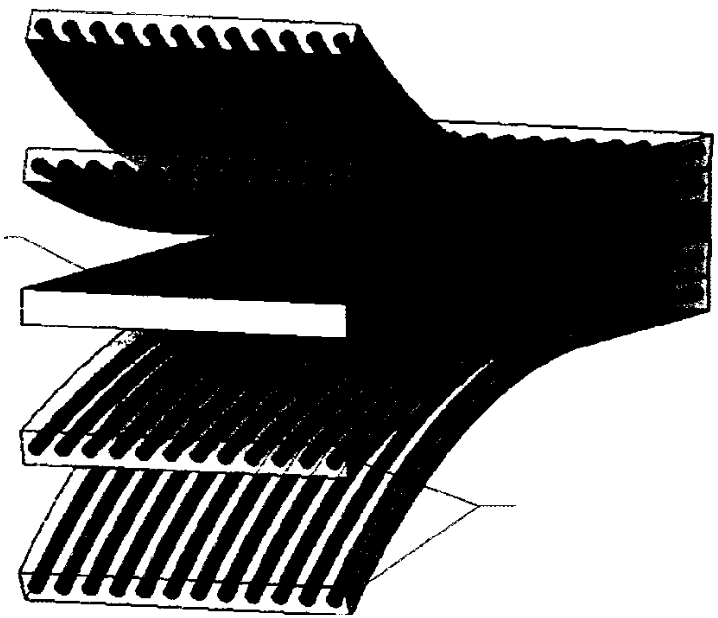


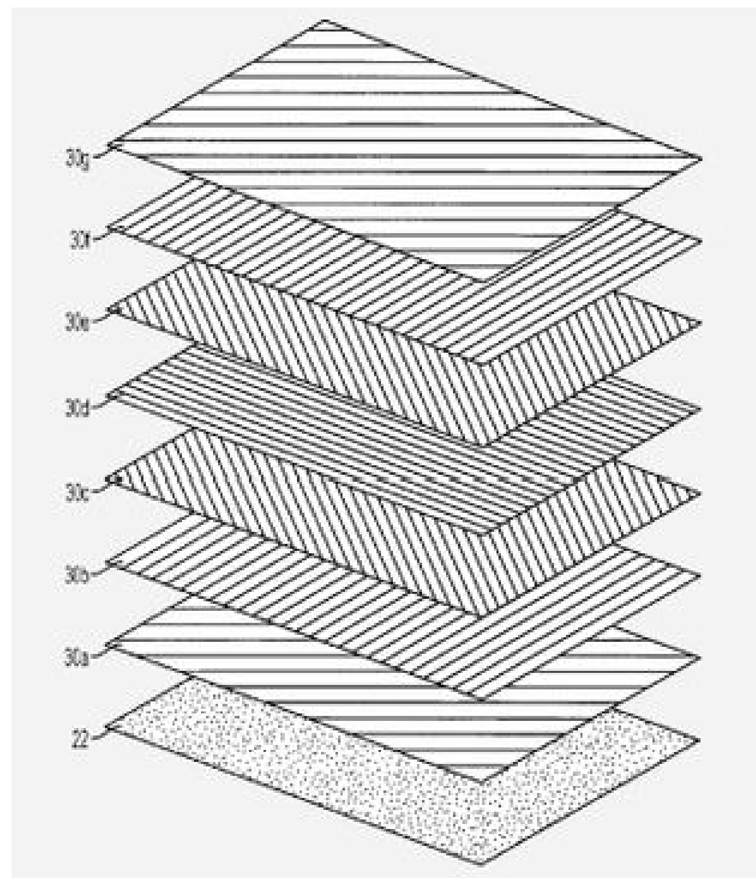
Materiais Compósitos

- Laminados

Em ângulo (angle-ply)

Unidericionais





II. Materiais Anisotrópicos

Teoria da Elasticidade

Estática/Pequenas Deformações/Material Linear

$$\nabla \sigma = 0 \qquad \sigma_{ij,j} = 0$$

$$\varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\sigma = c\,\varepsilon \qquad \sigma_{ij} = c_{ijkl}\,\varepsilon_{kl}$$

σ : tensor de tensões

u : vetor deslocamento

ε : tensor de deformações

c : tensor de elasticidade

Simetrias do Tensor de Elasticidade

Simetrias Menores

$\sigma_{ij} = \sigma_{ji} \Rightarrow c_{ijkl} = c_{jikl}$ (27 componentes repetidas)

$\varepsilon_{kl} = \varepsilon_{lk} \Rightarrow c_{ijkl} = c_{ijlk}$ (18 componentes repetidas)

$\downarrow$

36 constantes independentes

Simetrias do Tensor de Elasticidade

Densidade de Energia de Deformação Elástica

$$v(\varepsilon) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\varepsilon_{ij}c_{ijkl}\varepsilon_{kl} > 0$$ (c_{ijkl} é positivo—definido)

$$\sigma_{ij} = \frac{\partial v(\varepsilon)}{\partial \varepsilon_{ij}} = c_{ijkl}\varepsilon_{kl}\ , \quad c_{ijkl} = \frac{\partial^2 v(\varepsilon)}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}$$

$c_{ijkl} = c_{klij}$ (15 componentes repetidas)

21 constantes independentes

Simetrias do Tensor de Elasticidade

Notação de Voigt: $\sigma_I = c_{IJ}\,\varepsilon_J \quad \left(\varepsilon_I = s_{IJ}\,\sigma_J\right)$

ij		I
11	11	1
22	22	2
33	33	3
23	32	4
13	31	5
12	21	6

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

Simetrias do Tensor de Elasticidade

Notação de Voigt: $\sigma_I = c_{IJ}\,\varepsilon_J$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1113} & c_{1112} \\ & c_{2222} & c_{2233} & c_{2223} & c_{2213} & c_{2212} \\ & & c_{3333} & c_{3323} & c_{3313} & c_{3312} \\ & & & c_{2323} & c_{2313} & c_{2312} \\ & & & & c_{1313} & c_{1312} \\ & & & & & c_{1212} \end{bmatrix}$$

Simetrias do Tensor de Elasticidade

Notação de Voigt: $\sigma_I = c_{IJ}\,\varepsilon_J$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

Sistemas e Classes de Simetria

Sistema Triclínico

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix}$$

Ref.: Auld, B. A., *Acoustic Fields and Waves in Solids*, Vol I, 2nd ed., Krieger Publishing Company, 1990

Sistemas e Classes de Simetria

Sistema Monoclínico

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & c_{15} & 0 \\ & c_{22} & c_{23} & 0 & c_{25} & 0 \\ & & c_{33} & 0 & c_{35} & 0 \\ & & & c_{44} & 0 & c_{46} \\ & & & & c_{55} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Sistemas e Classes de Simetria

Sistema Ortorrômbico

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Sistemas e Classes de Simetria

Sistema Tetragonal Classes $4, \bar{4}, 4/m$

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ & c_{11} & c_{13} & 0 & 0 & -c_{16} \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Sistemas e Classes de Simetria

Sistema Tetragonal Classes $4mm,\ 422,\ \bar{4}2m,\ 4/mmm$

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Sistemas e Classes de Simetria

Sistema Trigonal (Romboédrico) Classes $3, \bar{3}$

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & -c_{25} & 0 \\ & c_{11} & c_{13} & c_{14} & c_{25} & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & c_{25} \\ & & & & c_{44} & c_{14} \\ & & & & & (c_{11}-c_{12})/2 \end{bmatrix}$$

Sistemas e Classes de Simetria

quartzo

Sistema Trigonal (Romboédrico) Classes $32, 3m, \bar{3}m$

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & c_{14} \\ & & & & & (c_{11}-c_{12})/2 \end{bmatrix}$$

Sistemas e Classes de Simetria

rochas, compósitos carbono-epóxi

Sistema Hexagonal (*Isotropia Transversal*)

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & (c_{11} - c_{12})/2 \end{bmatrix}$$

Sistemas e Classes de Simetria

Sistema Cúbico

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{11} & c_{12} & 0 & 0 & 0 \\ & & c_{11} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{44} \end{bmatrix}$$

Sistemas e Classes de Simetria

Sistema Isotrópico

$$c = \begin{bmatrix} c_{12}+2c_{44} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{12}+2c_{44} & c_{12} & 0 & 0 & 0 \\ & & c_{12}+2c_{44} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{44} \end{bmatrix}$$

Cosntantes de Lamé:

$$c_{12} = \lambda = E\nu/(1+\nu)(1-2\nu), \quad c_{44} = \mu = E/2(1+\nu)$$

Ref.: Auld, B. A., *Acoustic Fields and Waves in Solids*, Vol I, 2nd ed., Krieger Publishing Company, 1990

MECHANICAL PROPERTIES A.1 Crystal Symmetry Class and Mass Density

Material	Chemical Formula	Symmetry Class	Density (kg/m^3)§
INSULATORS			
Ammonium Dihydrogen Phosphate (ADP)†	$NH_4H_2PO_4$	Tetr. $\bar{4}2m$	1803
Barium Fluoride	BaF_2	Cubic *m3m*	4886
Barium Sodium Niobate†	$Ba_2NaNb_5O_{15}$	Orth. *2mm*	5300
Barium Titanate†	$BaTiO_3$	Tetr. *4mm*	6020
Barium Titanate†	$BaTiO_3$	Uniaxial‖	5700
Bismuth Germanate†	$Bi_4Ge_3O_{12}$	Cubic $\bar{4}3m$	7095
Bismuth Germanium Oxide†	$Bi_{12}GeO_{20}$	Cubic *23*	9200
Calcium Molybdate	$CaMoO_4$	Tetr. *4/m*	4255
Diamond	C	Cubic *m3m*	3512
Europium Iron Garnet‡	$Eu_3Fe_5O_{12}$	Cubic *m3m*	6280
Fused Silica	SiO_2	Isotropic	2200
Glass—Heavy Silicate Flint	—	Isotropic	3879
Glass—Light Borate Crown	—	Isotropic	2243
Glass—Pyrex	—	Isotropic	2320
Glass—T-40	—	Isotropic	3390
Iodic Acid†	HIO_3	Orth. *222*	4629
Lead Molybdate	$PbMoO_4$	Tetr. *4/m*	6950
Lead Titanate-Zirconate (PZT-2)†	—	Uniaxial‖	7600
Lead Titanate-Zirconate (PZT-5H)†	—	Uniaxial‖	7500
Lithium Fluoride	LiF	Cubic *m3m*	2601
Lithium Gallate†	$LiGaO_2$	Orth. *2mm*	4187
Lithium Niobate†	$LiNbO_3$	Trig. *3m*	4700
Lithium Tantalate†	$LiTaO_3$	Trig. *3m*	7450
Lucite	—	Isotropic	1182
Magnesium Oxide	MgO	Cubic *m3m*	3650
Polyethylene	—	Isotropic	900
Polystyrene	—	Isotropic	1056
Potassium Dihydrogen Phosphate (KDP)†	KH_2PO_4	Tetr. $\bar{4}2m$	2338
Quartz†	SiO_2	Trig. *32*	2651
Rochelle Salt†	$NaKC_4H_4O_6 \cdot 4H_2O$	Orth. *222*	1767
Rutile	TiO_2	Tetr. *4/mmm*	4260
Sapphire	Al_2O_3	Trig. $\bar{3}m$	3986
Sodium Fluoride	NaF	Cubic *m3m*	2790
Strontium Titanate	$SrTiO_3$	Cubic *m3m*	5110
Yttrium Aluminum Garnet (YAG)	$Y_3Al_5O_{12}$	Cubic *m3m*	4550
Yttrium Gallium Garnet (YGaG)	$Y_3Ga_5O_{12}$	Cubic *m3m*	5790
Yttrium Iron Garnet (YIG)‡	$Y_3Fe_5O_{12}$	Cubic *m3m*	5170
SEMICONDUCTORS			
Beryllium Oxide†	BeO	Hex. *6mm*	3009
Cadmium Selenide†	CdSe	Hex. *6mm*	5684
Cadmium Sulfide†	CdS	Hex. *6mm*	4820
Gallium Arsenide†	GaAs	Cubic $\bar{4}3m$	5307
Gallium Phosphide†	GaP	Cubic $\bar{4}3m$	4130
Germanium	Ge	Cubic *m3m*	5322
Indium Antimonide†	InSb	Cubic $\bar{4}3m$	5770
Indium Arsenide†	InAs	Cubic $\bar{4}3m$	5672
Indium Phosphide†	InP	Cubic $\bar{4}3m$	4787
Silicon	Si	Cubic *m3m*	2332
Tellurium Dioxide†	TeO_2	Tetr. *422*	5990
Zinc Oxide†	ZnO	Hex. *6mm*	5680
Zinc Sulfide†	ZnS	Hex. *6mm*	3980
METALS			
Aluminum	Al	Cubic *m3m*	2695
Gold	Au	Cubic *m3m*	19,300
Indium	In	Tetr. *4/mmm*	7300
Iron‡	Fe	Cubic *m3m*	7870
Nickel‡	Ni	Cubic *m3m*	8905
Silver	Ag	Cubic *m3m*	10,490
Tellurium	Te	Trig. *32*	6260
Titanium	Ti	Hex. *6/mmm*	4500
Tungsten	W	Cubic *m3m*	19,200

† Piezoelectric.
‡ Ferromagnets and ferrimagnets.
§ Units and conversion ratios on back endpaper.
‖ Poled ceramic.

Ref.: Auld, B. A., *Acoustic Fields and Waves in Solids*, Vol I, 2nd ed., Krieger Publishing Company, 1990

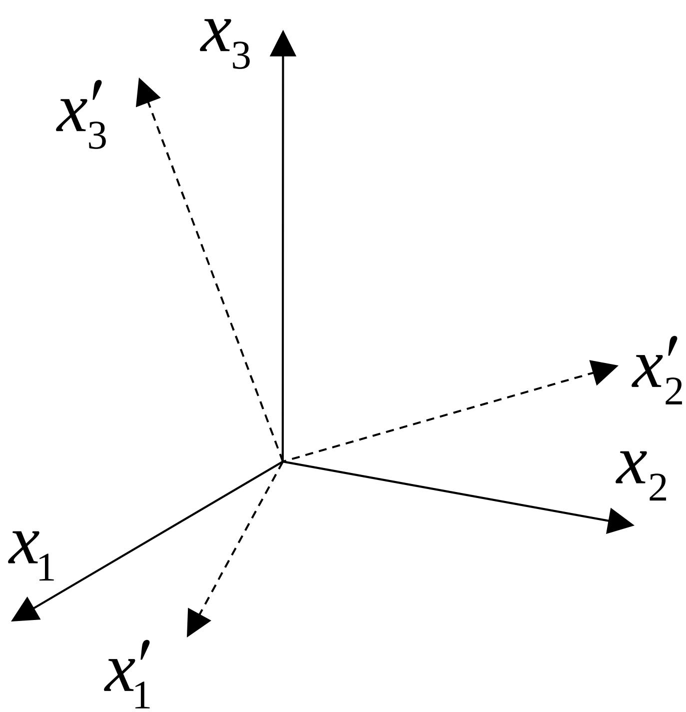
MECHANICAL PROPERTIES A.5 Stiffness Constants c_{IJ}

Material	Stiffness (10^{10} newton/m^2)§											
	c_{11}	c_{22}	c_{33}	c_{44}	c_{55}	c_{66}	c_{12}	c_{13}	c_{14}	c_{16}	c_{23}	c_{25}
INSULATORS												
Ammonium Dihydrogen Phosphate (ADP)†	6.76		3.38	0.867		0.608	0.59	2.0				
Barium Fluoride	9.20			2.57			4.18					
Barium Sodium Niobate†	23.9	24.7	13.5	6.5	6.6	7.6	10.4	5.0			5.2	
Barium Titanate†	27.5		16.5	5.43		11.3	17.9	15.1				
Barium Titanate†·‖	15.0		14.6	4.4			6.6	6.6				
Bismuth Germanate†	11.58			4.36			2.70					
Bismuth Germanium Oxide†	12.80			2.55			3.05					
Calcium Molybdate	14.47		12.65	3.69		4.51	6.64	4.46		1.34		
Diamond	102			49.2			25					
Europium Iron Garnet‡	25.1			7.62			10.70					
Fused Silica	7.85			3.12								
Glass—Heavy Silicate, Flint	6.13			2.18								
Glass—Light Borate, Crown	5.82			1.81								
Glass—Pyrex	7.3			2.5								
Glass—T-40	6.30			2.26								
Iodic Acid†	3.03	5.45	4.36	1.84	2.19	1.74	1.19	1.17			0.55	
Lead Molybdate	10.92		9.17	2.67		3.37	6.83	5.28		1.36		

Ref.: Auld, B. A., *Acoustic Fields and Waves in Solids*,
Vol I, 2nd ed., Krieger Publishing Company, 1990

Lead Titanate-Zirconate (PZT-2)†·‖	13.5		11.3	2.22			6.79	6.81		
Lead Titanate-Zirconate (PZT-5H)†·‖	12.6		11.7	2.30			7.95	8.41		
Lithium Fluoride	11.12			6.28			4.20			
Lithium Gallate†	—	—	14.7	5.42	4.45	—	—	—		—
Lithium Niobate†	20.3		24.5	6.0			5.3	7.5	0.9	
Lithium Tantalate†	23.3		27.5	9.4			4.7	8.0	−1.1	
Lucite	0.848			0.143						
Magnesium Oxide	28.6			14.8			8.7			
Polyethylene	0.340			0.026						
Polystyrene	0.58			0.12						
Potassium Dihydrogen Phosphate (KDP)†	7.85		7.63	1.23		0.61	3.2	3.87		
Quartz†	8.674		10.72	5.794			0.699	1.191	+1.791	
Rochelle Salt†	2.8	4.14	3.94	0.666	0.285	0.96	1.74	1.50		1.97
Rutile	26.60		46.99	12.39		18.86	17.33	13.62		
Sapphire	49.4		49.6	14.5			15.8	11.4	−2.3	
Sodium Fluoride	9.7			2.81			2.44			
Strontium Titanate	31.8			12.4			10.2			
Yttrium Aluminum Garnet (YAG)	33.32			11.50			11.07			
Yttrium Gallium Garnet	29.03			9.55			11.73			
Yttrium Iron Garnet (YIG)‡	26.8			7.66			11.06			

Transformação de Coordenadas



$$\sigma'_{pq} = a_{pi} a_{qj} \sigma_{ij}$$

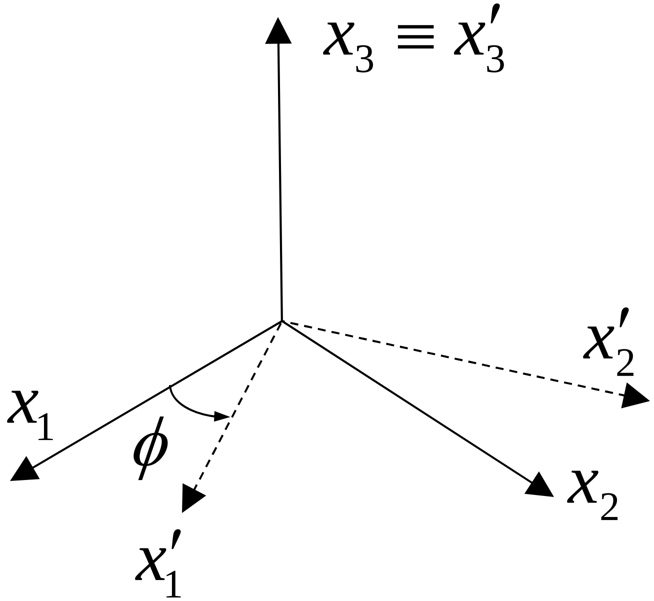
$$\varepsilon'_{pq} = a_{pi} a_{qj} \varepsilon_{ij}$$

$$c'_{pqrs} = a_{pi} a_{qj} a_{rk} a_{sl} c_{ijkl}$$

a_{pi} é uma matriz ortogonal de transformação (rotação)

Transformação de Coordenadas

Ex.: Rotação de um ângulo ϕ em torno do eixo x_3



$$[\sigma'] = [a][\sigma][a]^T$$

$$[a] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\sigma'] = \begin{bmatrix} \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\phi + \sigma_{12}\sin 2\phi & -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\phi + \sigma_{12}\cos 2\phi & \sigma_{13}\cos\phi + \sigma_{23}\sin\phi \\ -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\phi + \sigma_{12}\cos 2\phi & \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\phi - \sigma_{12}\sin 2\phi & -\sigma_{13}\sin\phi + \sigma_{23}\cos\phi \\ \sigma_{13}\cos\phi + \sigma_{23}\sin\phi & -\sigma_{13}\sin\phi + \sigma_{23}\cos\phi & \sigma_{33} \end{bmatrix}$$

Transformação de Coordenadas

Utilizando a notação reduzida (*notação de Voigt)*

$$\sigma'_{pq} = a_{pi} a_{qj} \sigma_{ij}$$

$$\varepsilon'_{pq} = a_{pi} a_{qj} \varepsilon_{ij}$$

$$c'_{pqrs} = a_{pi} a_{qj} a_{rk} a_{sl} c_{ijkl}$$

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad [a]^{-1} = [a]^T$$

Transformação de Coordenadas

Utilizando a notação reduzida (*notação de Voigt)*

$$\sigma'_P = M_{PI}\sigma_I$$

$$\begin{bmatrix} \sigma'_1 \\ \sigma'_2 \\ \sigma'_3 \\ \sigma'_4 \\ \sigma'_5 \\ \sigma'_6 \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{13}a_{11} & 2a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{23}a_{21} & 2a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{33}a_{31} & 2a_{31}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33}+a_{23}a_{32} & a_{21}a_{33}+a_{23}a_{31} & a_{22}a_{31}+a_{21}a_{32} \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & a_{12}a_{33}+a_{13}a_{32} & a_{13}a_{31}+a_{11}a_{33} & a_{11}a_{32}+a_{12}a_{31} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{12}a_{23}+a_{13}a_{22} & a_{13}a_{21}+a_{11}a_{23} & a_{11}a_{22}+a_{12}a_{21} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Transformação de Coordenadas

Utilizando a notação reduzida (*notação de Voigt)*

$$\varepsilon'_P = N_{PI}\varepsilon_I$$

$$\begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ \varepsilon'_3 \\ \varepsilon'_4 \\ \varepsilon'_5 \\ \varepsilon'_6 \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{12}a_{13} & a_{13}a_{11} & a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & a_{22}a_{23} & a_{23}a_{21} & a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & a_{32}a_{33} & a_{33}a_{31} & a_{31}a_{32} \\ 2a_{21}a_{31} & 2a_{22}a_{32} & 2a_{23}a_{33} & a_{22}a_{33}+a_{23}a_{32} & a_{21}a_{33}+a_{23}a_{31} & a_{22}a_{31}+a_{21}a_{32} \\ 2a_{31}a_{11} & 2a_{32}a_{12} & 2a_{33}a_{13} & a_{12}a_{33}+a_{13}a_{32} & a_{13}a_{31}+a_{11}a_{33} & a_{11}a_{32}+a_{12}a_{31} \\ 2a_{11}a_{21} & 2a_{12}a_{22} & 2a_{13}a_{23} & a_{12}a_{23}+a_{13}a_{22} & a_{13}a_{21}+a_{11}a_{23} & a_{11}a_{22}+a_{12}a_{21} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

Transformação de Coordenadas

Utilizando a notação reduzida (*notação de Voigt)*

$$\{\sigma'\} = [M]\{\sigma\}$$

$$\{\sigma'\} = [M][c]\{\varepsilon\} = [M][c][N]^{-1}\{\varepsilon'\}$$

$$\{\varepsilon'\} = [N]\{\varepsilon\}$$

$$\{\sigma'\} = [c']\{\varepsilon'\}$$

$$[c'] = [M][c][N]^{-1}$$

Transformação de Coordenadas

Utilizando a notação reduzida (*notação de Voigt)*

A matriz de transformação $[a]$ é ortogonal, logo pode-se mostrar que a matriz $[N]^{-1}$ é a transposta da matriz $[M]$

$$[N]^{-1} = [M]^T$$

portanto

$$[c'] = [M][c][M]^T$$

Transformação de Coordenadas

Utilizando a notação reduzida (*notação de Voigt)*

Analogamente, para a matriz de flexibilidade (*compliance*)

$$\{\varepsilon\} = [s]\{\sigma\}$$

A transformação é:

$$[s'] = [N][s][N]^T$$

e $\{\varepsilon'\} = [s']\{\sigma'\}$

Transformação de Coordenadas

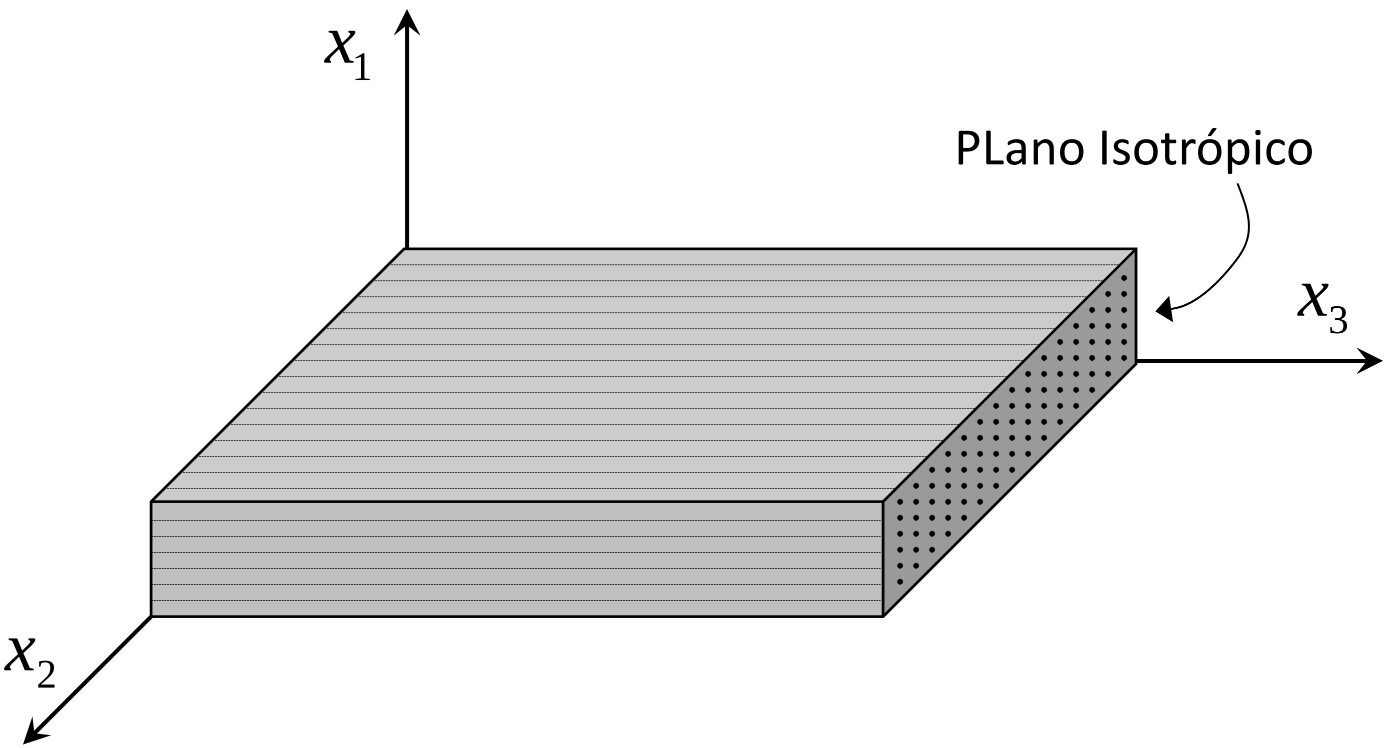
Exemplo: *Material Transversalmente Isotrópico*

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & (c_{11} - c_{12})/2 \end{bmatrix}$$

Transformação de Coordenadas

<u>Exemplo</u>: *Material Transversalmente Isotrópico*

Epóxi reforçado por fibras de carbono



Transformação de Coordenadas

<u>Exemplo</u>: *Material Transversalmente Isotrópico*
Epóxi reforçado por fibras de carbono (AS-3501)

$$[c]=\begin{bmatrix} 14.5 & 7.24 & 6.50 & 0 & 0 & 0 \\ & 14.5 & 6.50 & 0 & 0 & 0 \\ & & 161 & 0 & 0 & 0 \\ & & & 7.10 & 0 & 0 \\ & & & & 7.10 & 0 \\ & & & & & 3.63 \end{bmatrix} \text{ GPa}$$

Transformação de Coordenadas

Rotação de 45° em torno do eixo 1

$$[a]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$[M]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1 & 0 & 0 \\ 0 & 1/2 & 1/2 & -1 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Transformação de Coordenadas

Rotação de 45° em torno do eixo 1

$$[c'] = [M][c][M]^T$$

Acoplamento entre tensões normais e deformações cisalhantes

$$[c'] = \begin{bmatrix} 14.5 & 6.87 & 6.87 & -0.37 & 0 & 0 \\ & 54.2 & 40.0 & 36.6 & 0 & 0 \\ & & 54.2 & 36.6 & 0 & 0 \\ & & & 40.6 & 0 & 0 \\ & & & & 5.37 & 1.74 \\ & & & & & 5.37 \end{bmatrix} \text{GPa}$$

Transformação de Coordenadas

Rotação de 45° em torno do eixo 1

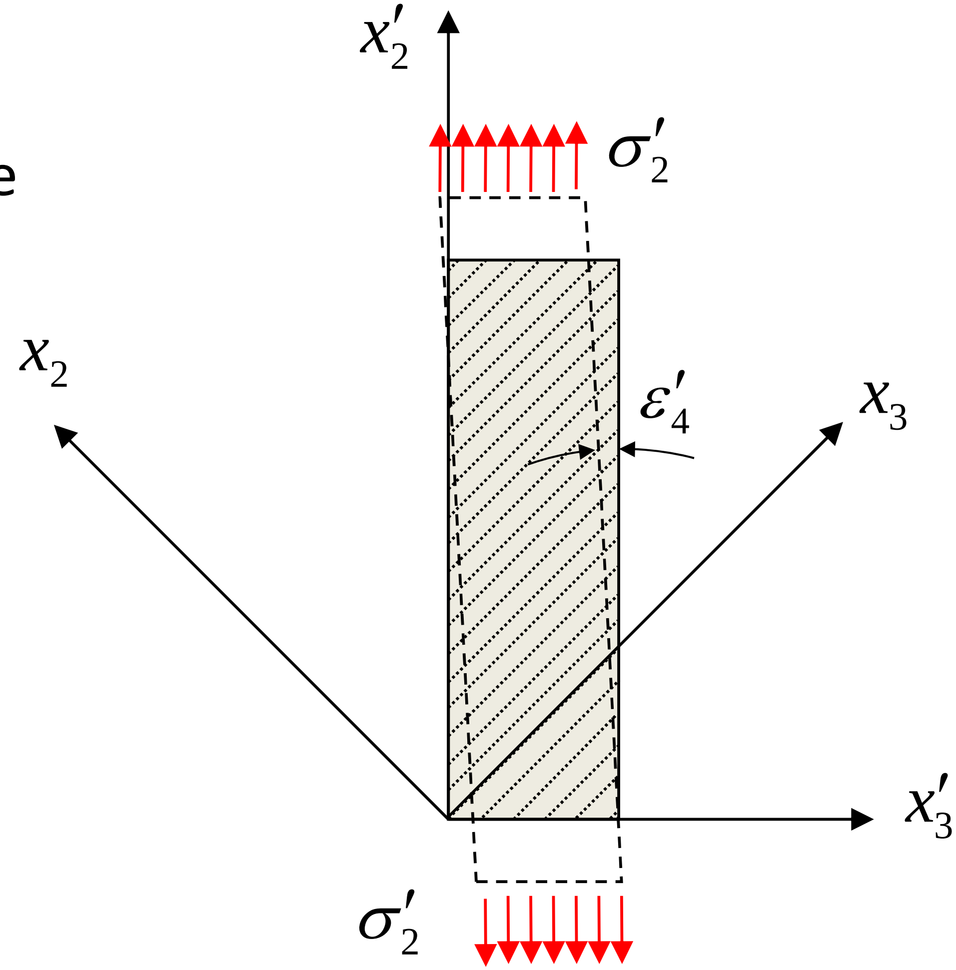
$$[s'] = [c']^{-1} = \begin{bmatrix} 92.4 & -23.6 & -23.6 & 43.4 & 0 & 0 \\ -23.6 & 59.0 & -11.5 & -43.0 & 0 & 0 \\ -23.6 & -11.5 & 59.0 & -43.0 & 0 & 0 \\ 43.4 & -43.0 & -43.0 & 103 & 0 & 0 \\ 0 & 0 & 0 & 0 & 208 & -67.3 \\ 0 & 0 & 0 & 0 & -67.3 & 208 \end{bmatrix} \times 10^{-12}\,\mathrm{Pa}^{-1}$$

Transformação de Coordenadas

Acoplamento entre tensão normal e deformação cisalhante (distorção)

$$
\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_3' \\ \varepsilon_4' \\ \varepsilon_5' \\ \varepsilon_6' \end{bmatrix}
=
\begin{bmatrix}
s_{11}' & s_{12}' & s_{12}' & s_{14}' & 0 & 0 \\
s_{12}' & s_{22}' & s_{23}' & s_{24}' & 0 & 0 \\
s_{12}' & s_{23}' & s_{33}' & s_{24}' & 0 & 0 \\
s_{14}' & s_{24}' & s_{24}' & s_{44}' & 0 & 0 \\
0 & 0 & 0 & 0 & s_{55}' & s_{56}' \\
0 & 0 & 0 & 0 & s_{56}' & s_{55}'
\end{bmatrix}
\begin{bmatrix} 0 \\ \sigma_2' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\varepsilon_4' = 2\varepsilon_{23}' = -43\times10^{-12}\ \sigma_2'$$



III. Propriedades (Micromecânica)

Material Isotrópico

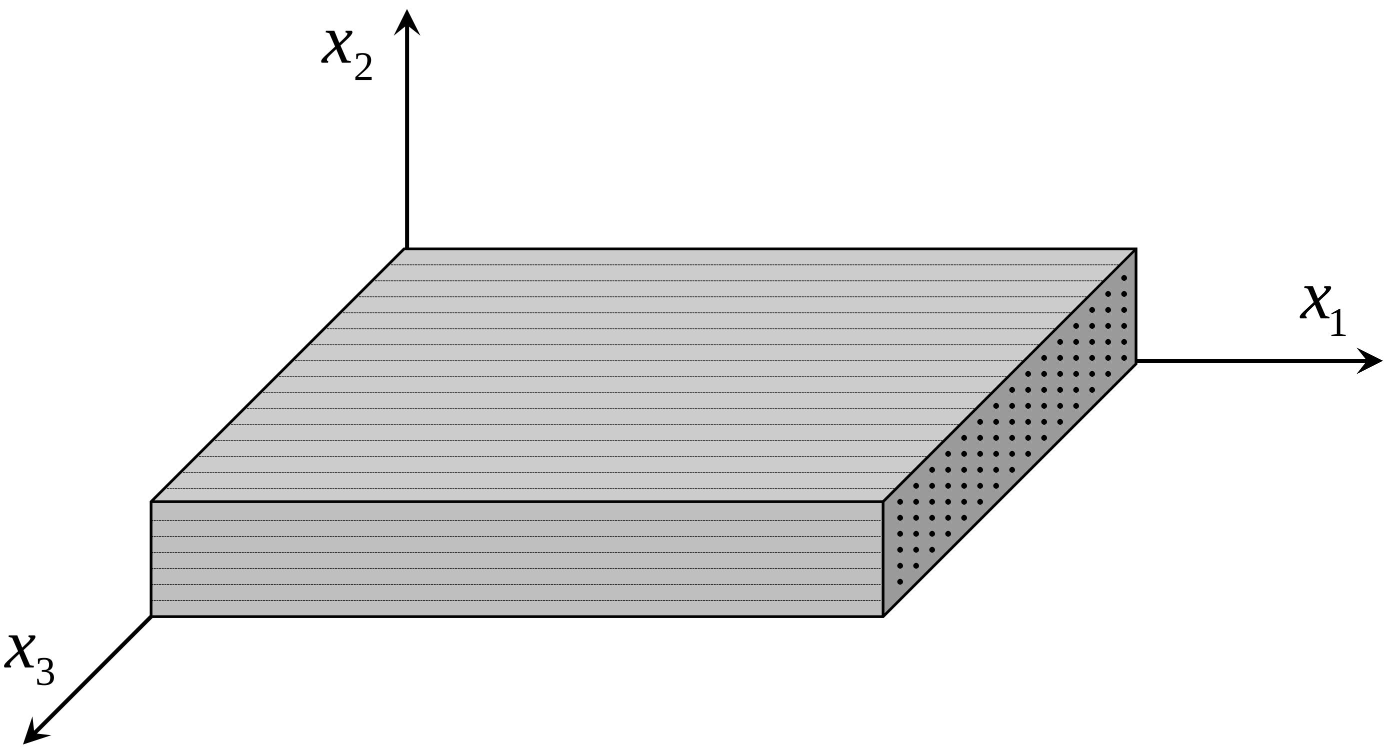
Relação Constitutiva
(utilizando as *constantes de engenharia*: E, ν)

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} =
\begin{bmatrix}
1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\
-\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\
-\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}
$$

$$G = E/2(1+\nu)$$

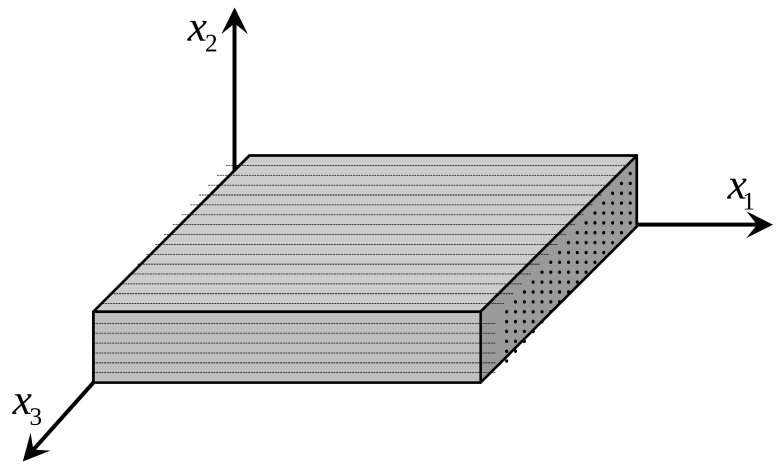
Material Ortotrópico (ortorrômbico)

Compósito reforçado por fibras



Material Ortotrópico (ortorrômbico)

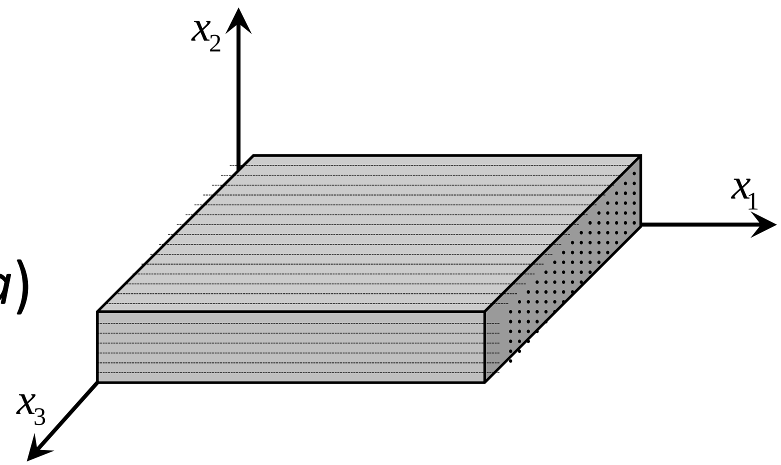
Relação constitutiva



$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Material Ortotrópico (ortorrômbico)

Relação Constitutiva
(utilizando as *constantes de engenharia*)

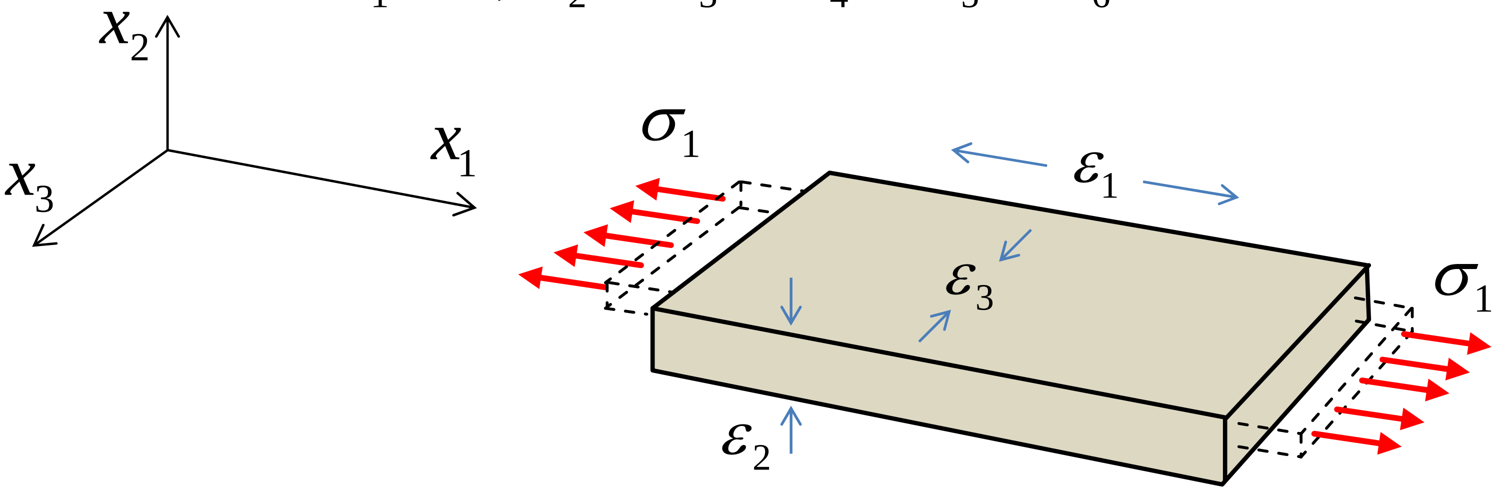


$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} 1/E_{11} & -\nu_{21}/E_{22} & -\nu_{31}/E_{33} & 0 & 0 & 0 \\ -\nu_{12}/E_{11} & 1/E_{22} & -\nu_{32}/E_{33} & 0 & 0 & 0 \\ -\nu_{13}/E_{11} & -\nu_{23}/E_{22} & 1/E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Determinação das Constantes Elásticas

Constantes elásticas do compósito podem ser obtidas a partir de ensaios de tração nas três ***direções principais de simetria*** do material

$$\sigma_1 \neq 0, \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0$$



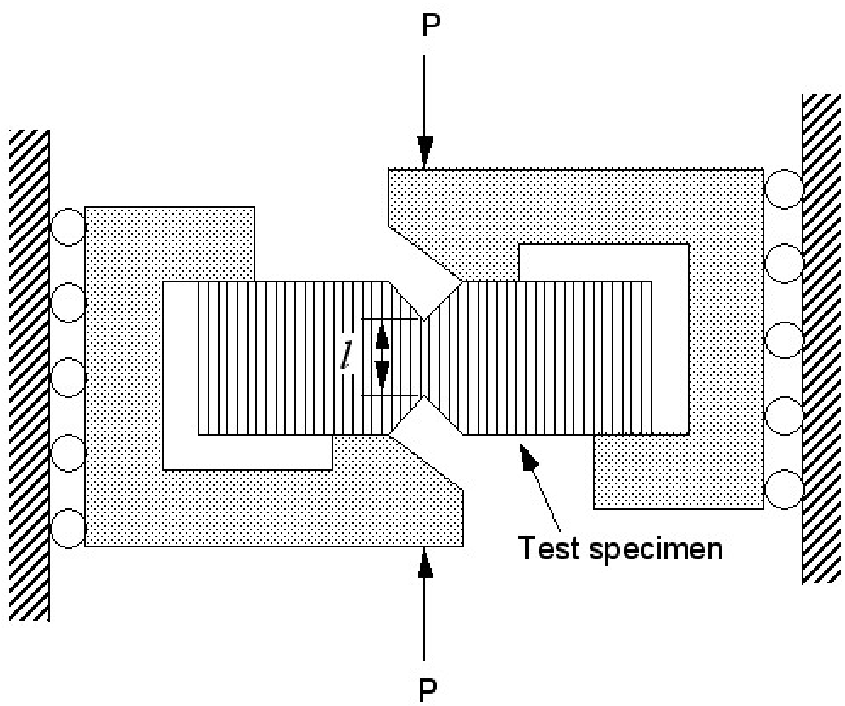
$$E_{11} = \sigma_1 / \varepsilon_1 \qquad \nu_{12} = -\varepsilon_2 / \varepsilon_1 \qquad \nu_{13} = -\varepsilon_3 / \varepsilon_1$$

Determinação das Constantes Elásticas

Cisalhamento

Corpo de Prova de Iosipescu (cisalhamento puro)

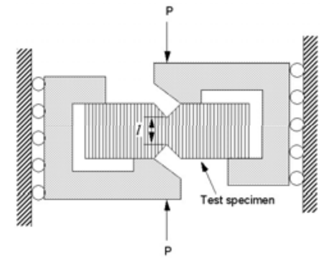
Norma ASTM D 7078-05: *V-Notch Rail Shear Test*

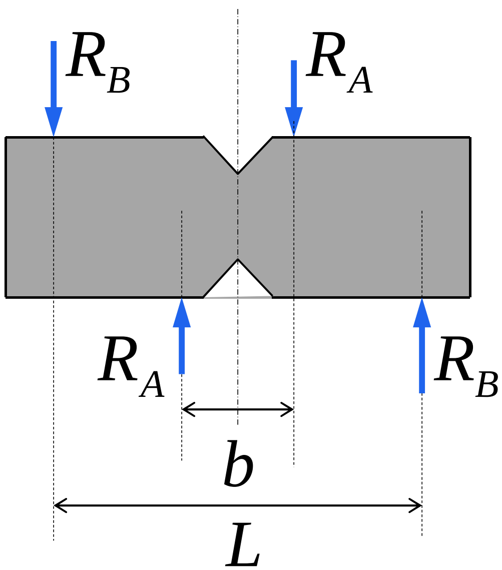


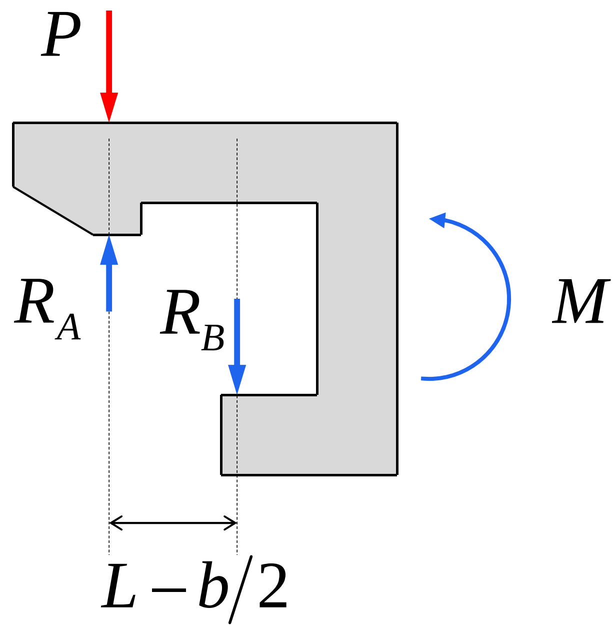
Determinação das Constantes Elásticas

Cisalhamento

Equilíbrio de forças e momentos no espécime







$$\boxed{R_A b - R_B L = 0}$$

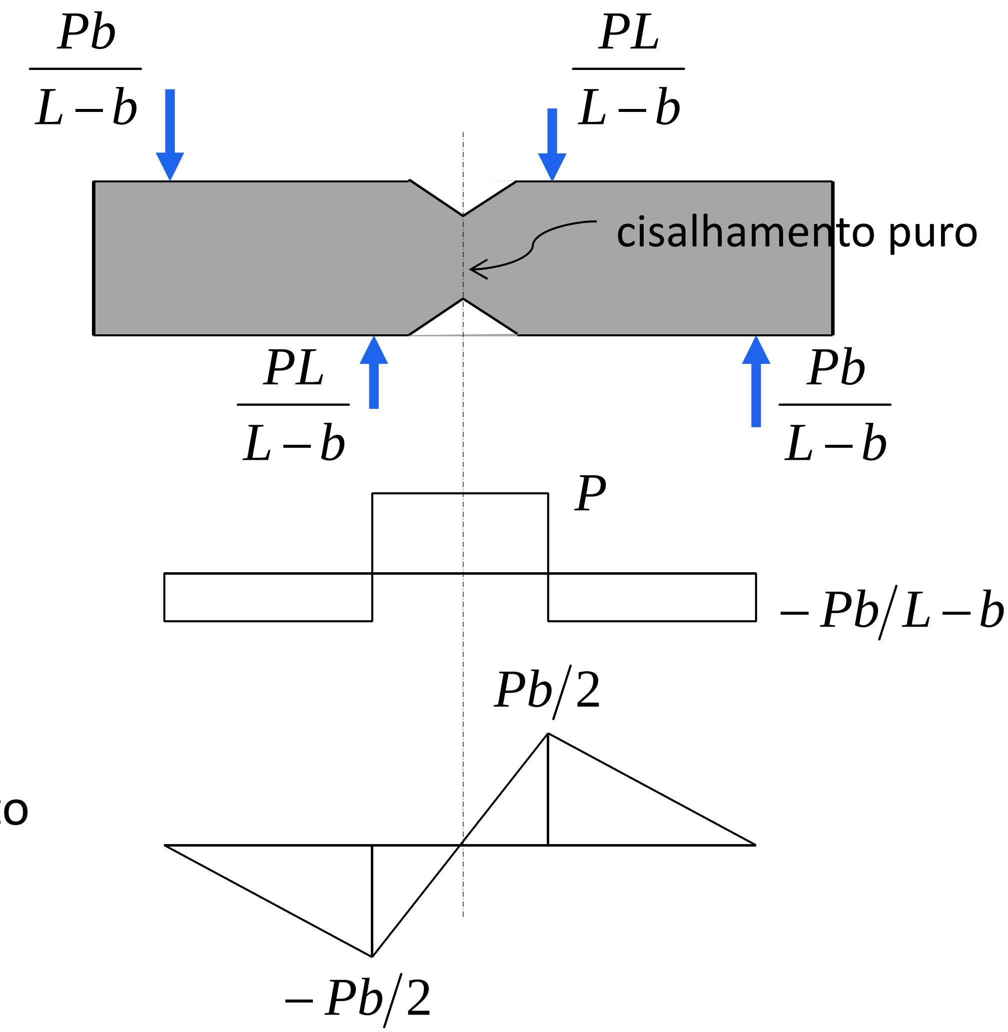
$$\boxed{R_A - R_B = P}$$

Determinação das Constantes Elásticas

Corpo de Prova

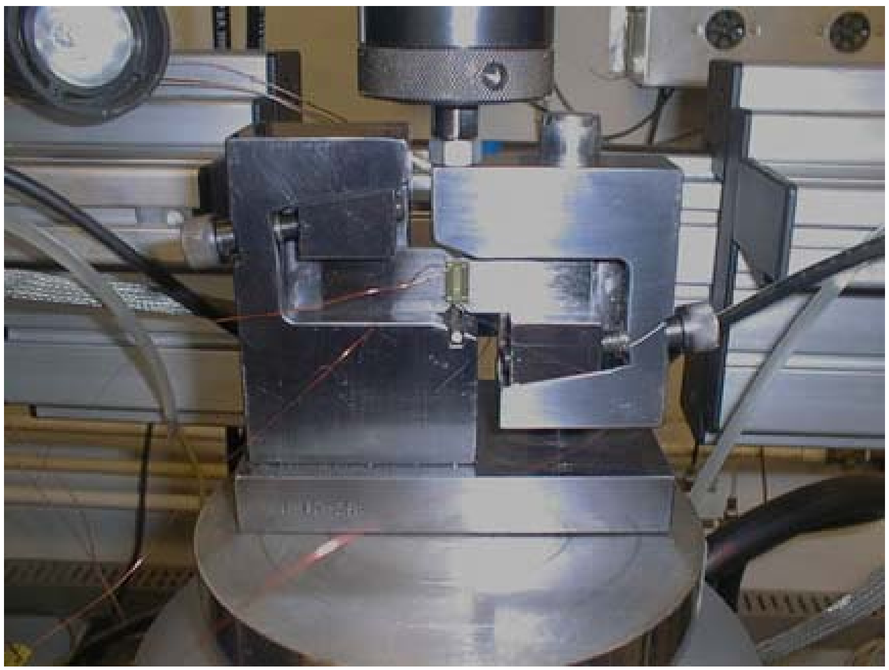
Diagrama de Esforço Cortante

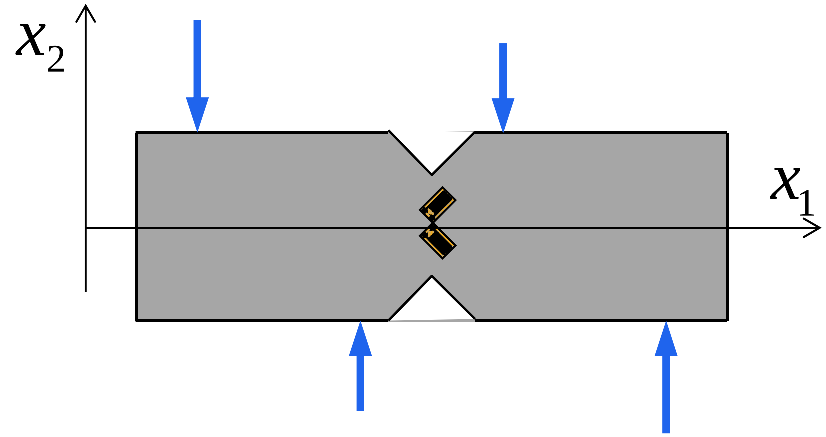
Diagrama de Momento Fletor

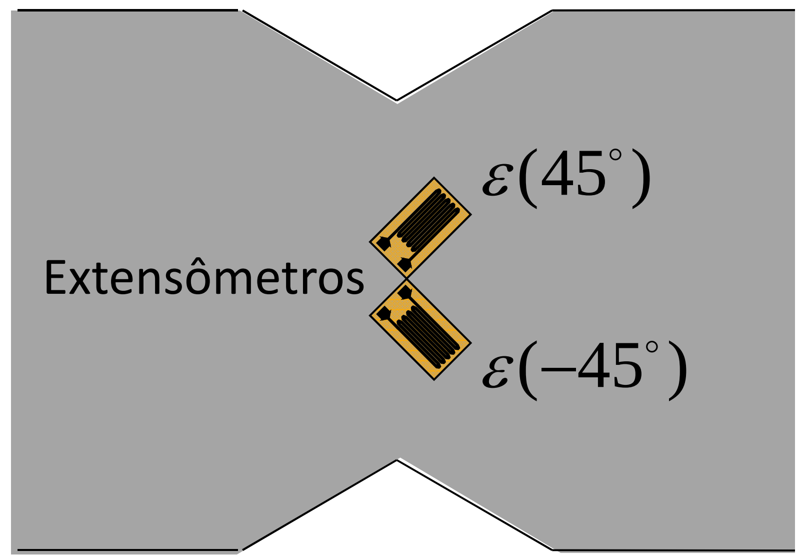


Determinação das Constantes Elásticas

Medida da Deformação Cisalhante



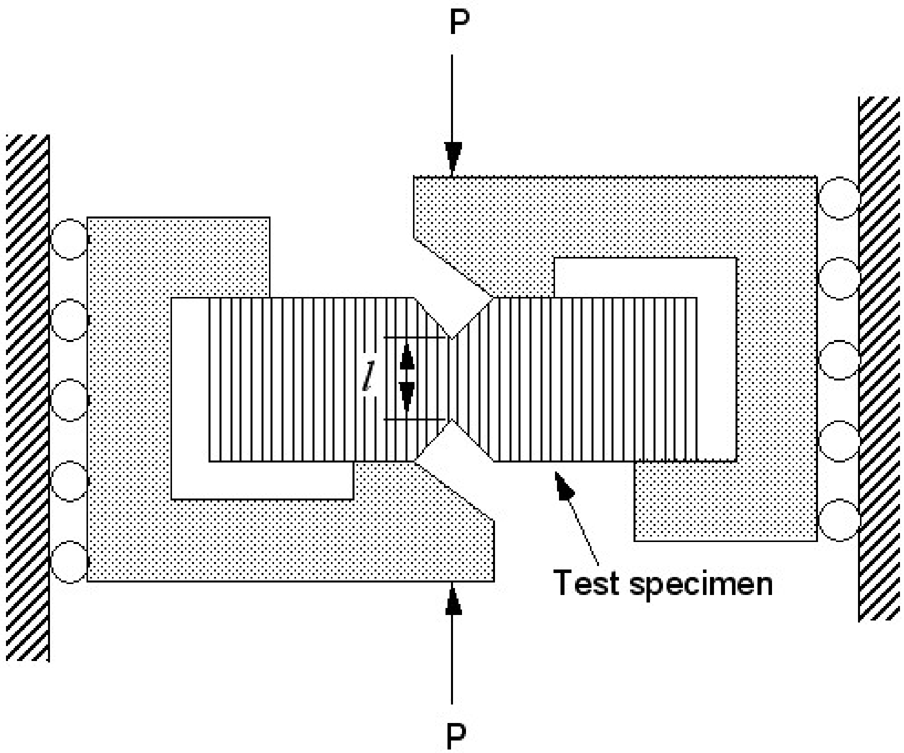




$$\boxed{\varepsilon_6 = 2\varepsilon_{12} = \varepsilon(45^\circ) - \varepsilon(-45^\circ)}$$

Determinação das Constantes Elásticas

Cisalhamento

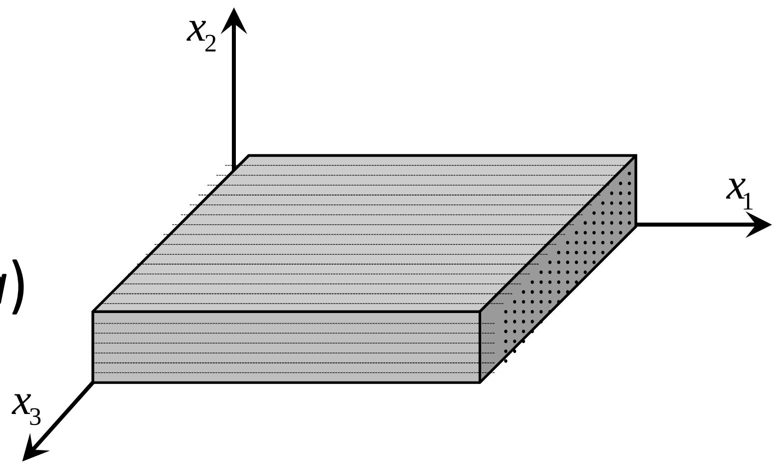


$$\sigma_6 = P/A$$

$$\varepsilon_6 = \varepsilon(45^\circ) - \varepsilon(-45^\circ)$$

$$\boxed{G_{12} = \sigma_6/\varepsilon_6}$$

Material Ortotrópico (ortorrômbico)

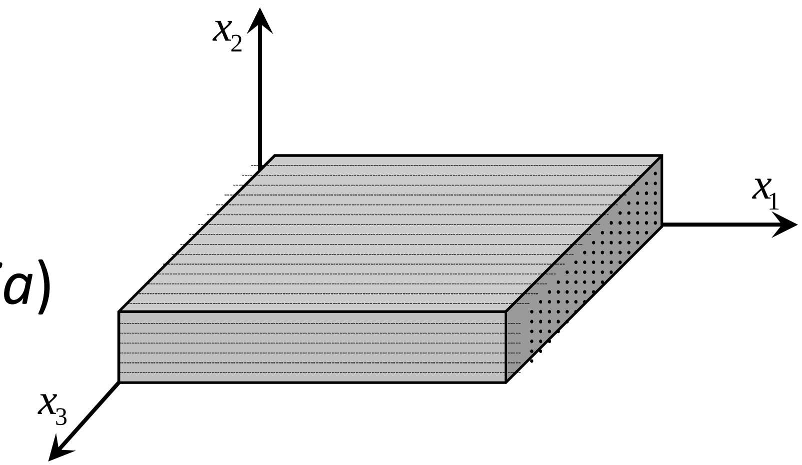


Relação Constitutiva
(utilizando as *constantes de engenharia*)

Simetria

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} 1/E_{11} & -\nu_{21}/E_{22} & -\nu_{31}/E_{33} & 0 & 0 & 0 \\ -\nu_{12}/E_{11} & 1/E_{22} & -\nu_{32}/E_{33} & 0 & 0 & 0 \\ -\nu_{13}/E_{11} & -\nu_{23}/E_{22} & 1/E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Material Ortotrópico (ortorrômbico)



Relação Constitutiva
(utilizando as *constantes de engenharia*)

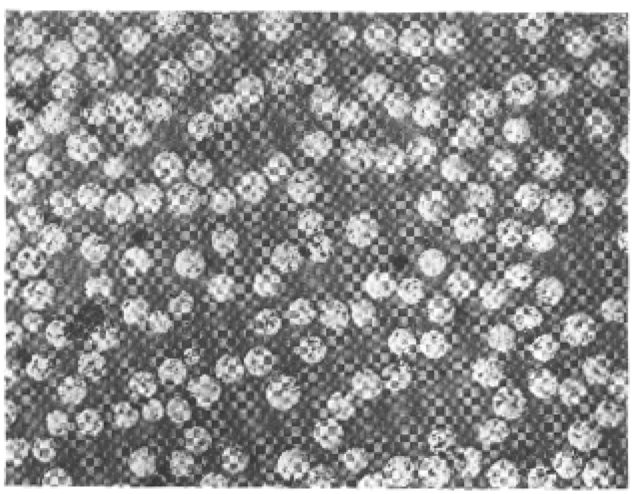
$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} =
\begin{bmatrix}
1/E_{11} & -\nu_{21}/E_{22} & -\nu_{31}/E_{33} & 0 & 0 & 0 \\
-\nu_{12}/E_{11} & 1/E_{22} & -\nu_{32}/E_{33} & 0 & 0 & 0 \\
-\nu_{13}/E_{11} & -\nu_{23}/E_{22} & 1/E_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_{12}
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}
$$

$$
\frac{\nu_{21}}{E_{22}} = \frac{\nu_{12}}{E_{11}}, \frac{\nu_{31}}{E_{33}} = \frac{\nu_{13}}{E_{11}}, \frac{\nu_{32}}{E_{33}} = \frac{\nu_{23}}{E_{22}}
$$

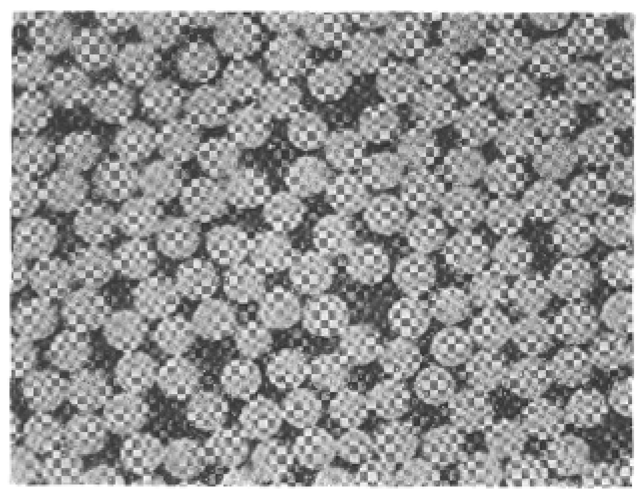
Determinação das Constantes Elásticas

Enfoque *Micromecânico*: determinação das constantes elásticas do compósito a partir das propriedades dos seus constituintes (matriz e fibras)

Seção transversal de compósitos reforçados por fibra unidirecionais



Fração volumétrica: 0,4



Fração volumétrica: 0,7

Determinação das Constantes Elásticas

Hipótese: fibra e matriz são de materiais homogêneos, isotrópicos, elásticos e lineares

E_f - Módulo de Elasticidade da Fibra

E_m - Módulo de Elasticidade da Matriz

G_f - Módulo de Cisalhamento da Fibra

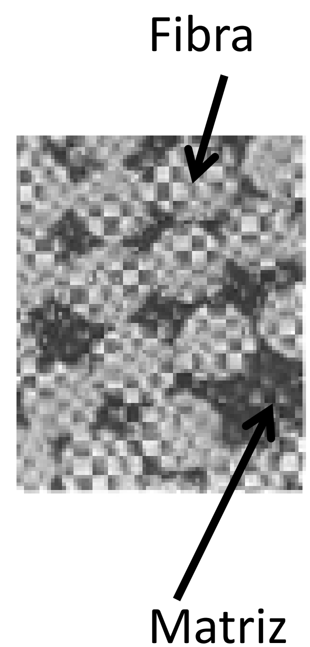
G_m - Módulo de Cisalhamento da Matriz

ν_f - Coeficiente de Poisson da Fibra

ν_m - Coeficiente de Poisson da Matriz

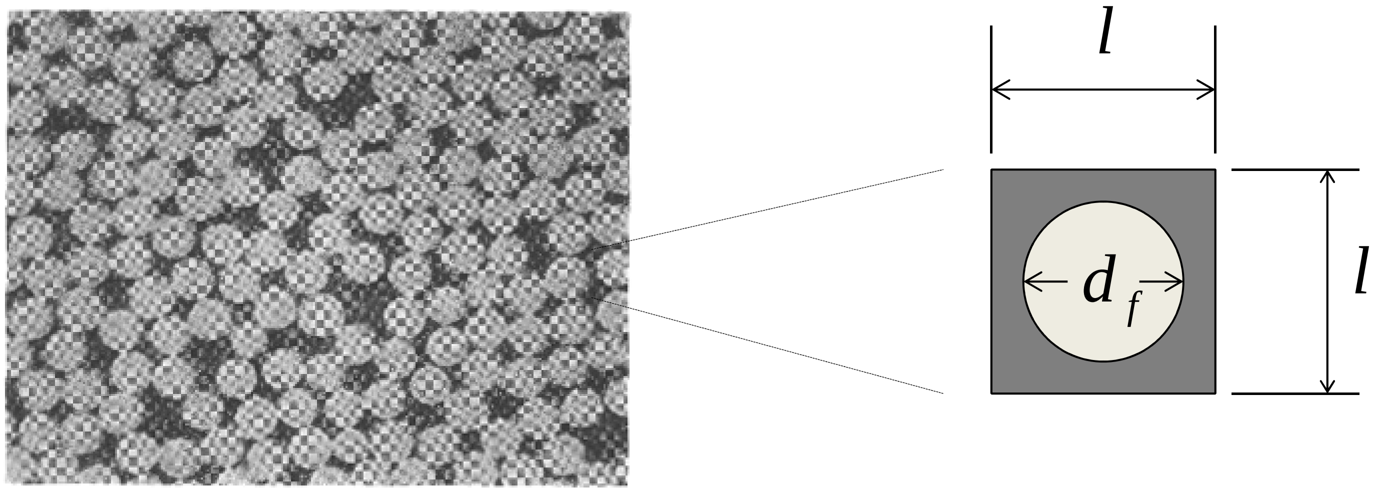
V_f - Fração Volumétrica da Fibra

V_m - Fração Volumétrica da Matriz $\left(V_m = 1 - V_f\right)$



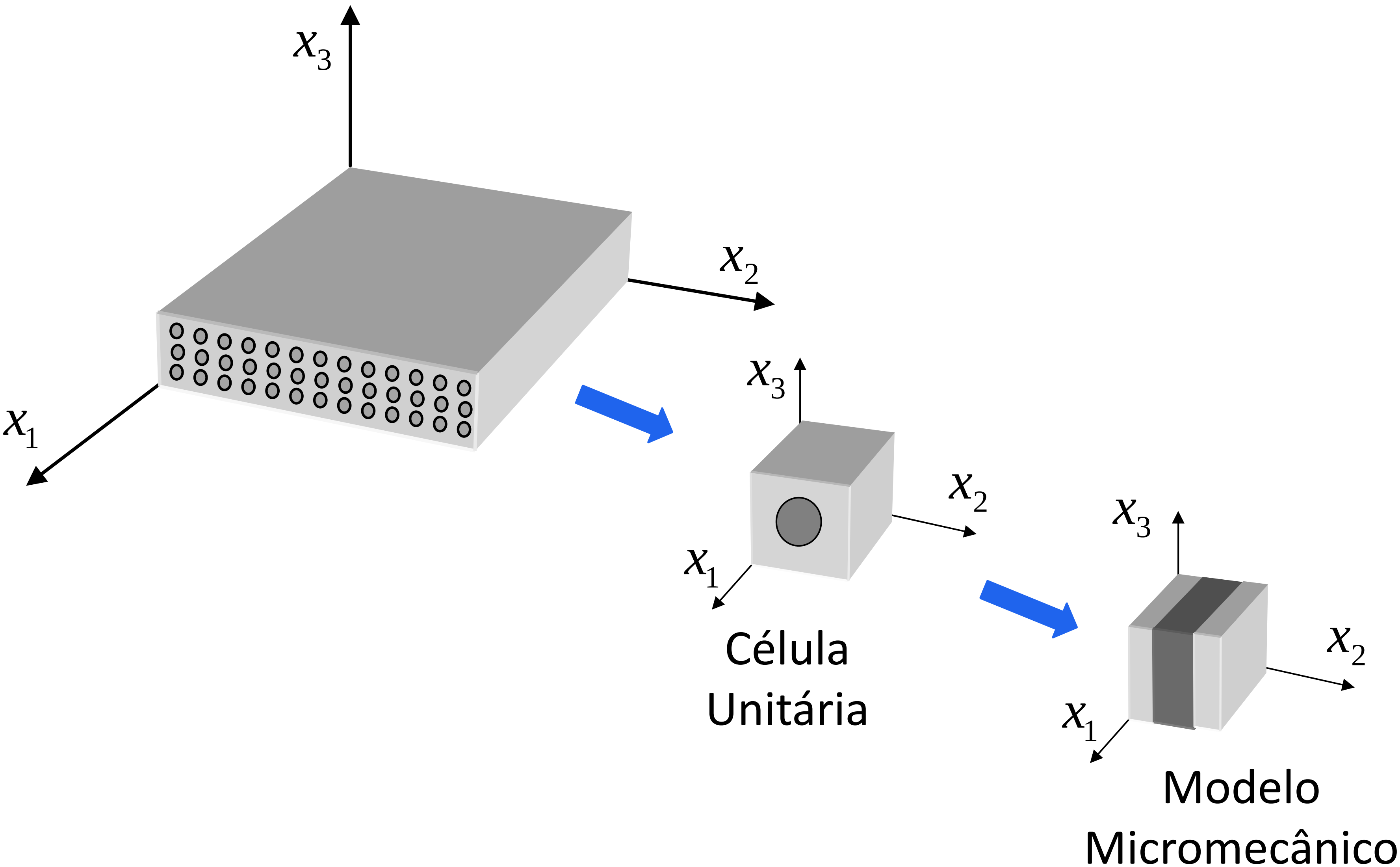
Determinação das Constantes Elásticas

$$V_f = \frac{\text{volume de fibra}}{\text{volume total}} = \frac{\pi d_f^2}{4l^2}$$



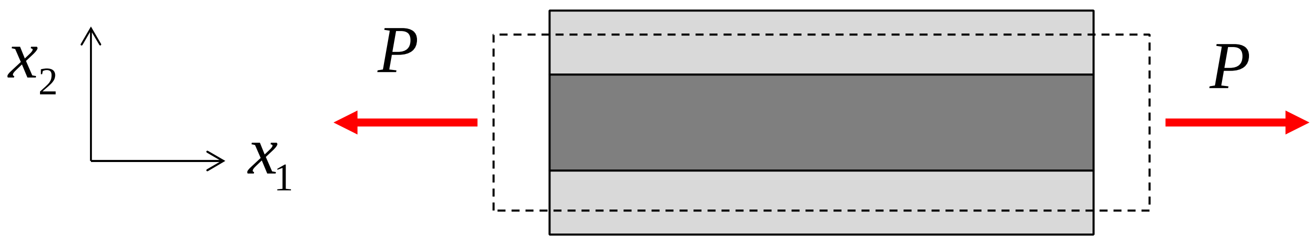
Máxima fração volumétrica: $d_f = l$ $\quad \max\{V_f\} = \dfrac{\pi}{4} = 0{,}785$

Determinação das Constantes Elásticas



Determinação das Constantes Elásticas

Cálculo de E_{11}



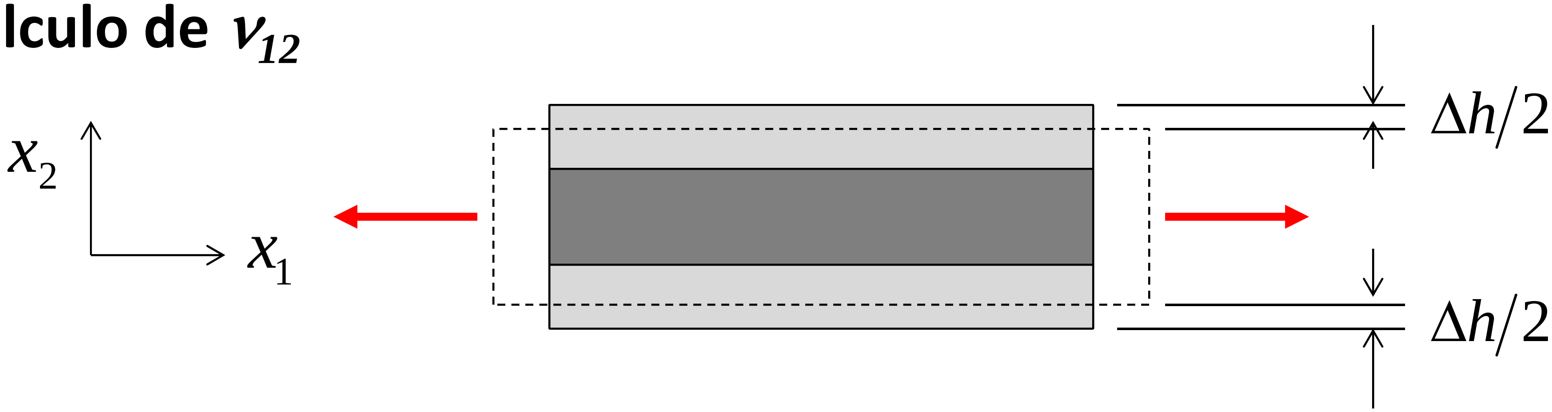
$$\left.\begin{array}{l} \sigma_1(A_m + A_f) = \sigma_1^f A_f + \sigma_1^m A_m \\ V_f = A_f/(A_m + A_f), V_m = A_m/(A_m + A_f) \end{array}\right\} \Rightarrow \quad \sigma_1 = \sigma_1^f V_f + \sigma_1^m V_m$$

$$\varepsilon_1 = \frac{\sigma_1}{E_{11}} = \frac{\sigma_1^f}{E_f} = \frac{\sigma_1^m}{E_m} \quad \Rightarrow \quad \frac{\sigma_1}{\varepsilon_1} = V_f \frac{\sigma_1^f}{\varepsilon_1^f} + V_f \frac{\sigma_1^f}{\varepsilon_1^f}$$

$$\boxed{E_{11} = V_f E_f + (1 - V_f) E_m}$$

Determinação das Constantes Elásticas

Cálculo de ν_{12}



$$h = h_f + h_m \qquad V_f = h_f/h,\ V_m = h_m/h$$

$$\Delta h = \Delta h_f + \Delta h_m$$

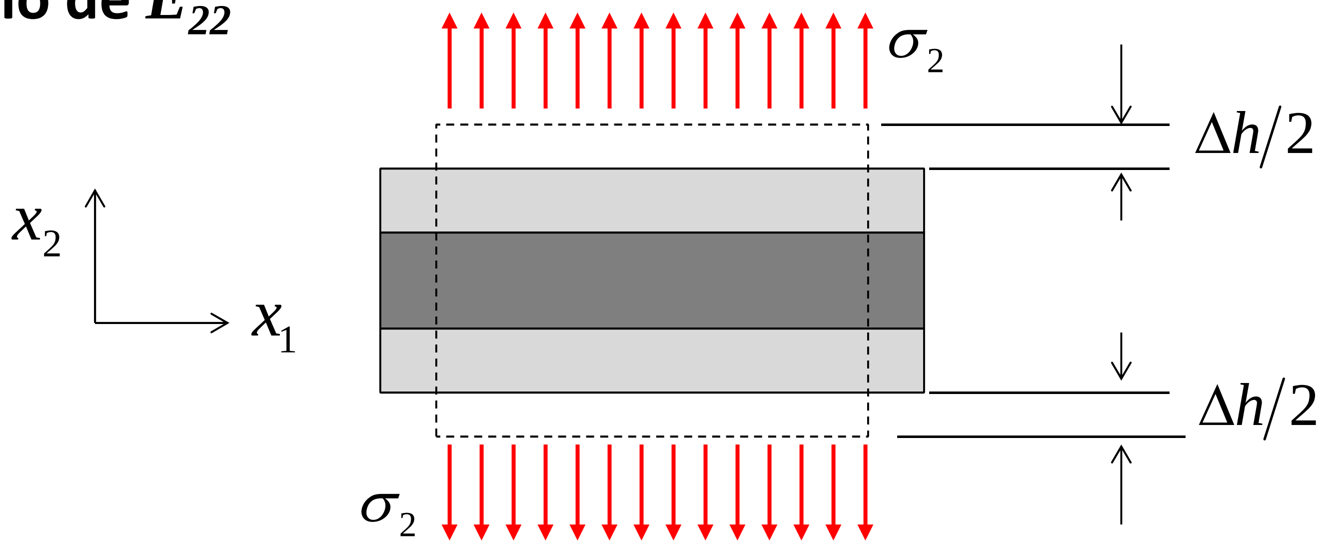
$$\varepsilon_2 = \frac{\Delta h_f + \Delta h_m}{h} = \frac{-\nu_f \varepsilon_1 h_f - \nu_m \varepsilon_1 h_m}{h}$$

$$\frac{\varepsilon_2}{\varepsilon_1} = -\nu_f \frac{h_f}{h} - \nu_m \frac{h_m}{h}$$

$$\boxed{\nu_{12} = \nu_f V_f + \nu_m (1 - V_f)}$$

Determinação das Constantes Elásticas

Cálculo de E_{22}



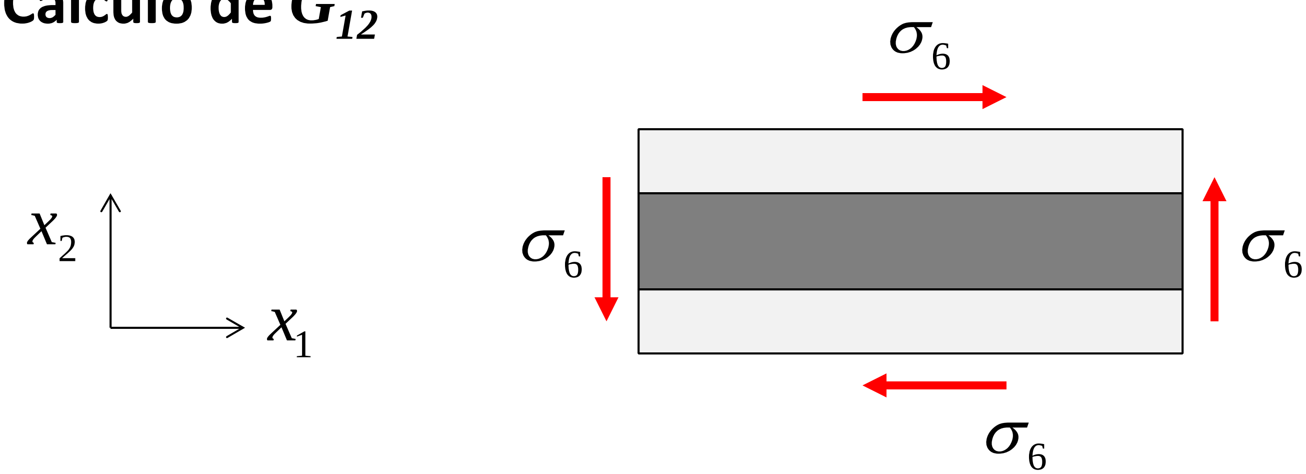
$$\Delta h = \varepsilon_2 h = \varepsilon_2^f h_f + \varepsilon_2^m h_m = \frac{\sigma_2}{E_f} h_f + \frac{\sigma_2}{E_m} h_m$$

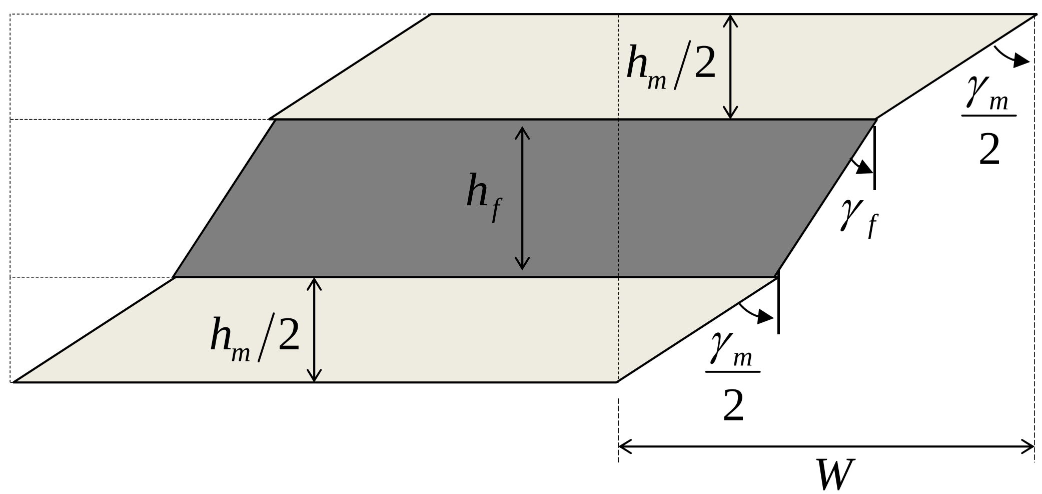
$$\varepsilon_2 = \frac{\Delta h}{h} = \sigma_2 \left(\frac{h_f / h}{E_f} + \frac{h_m / h}{E_m} \right)$$

$$\boxed{E_{22} = \frac{E_f E_m}{E_f (1 - V_f) + E_m V_m}}$$

Determinação das Constantes Elásticas

Cálculo de G_{12}





$$W = \gamma_f h_f + \gamma_m h_m$$

Determinação das Constantes Elásticas

Cálculo de G_{12}

$$W = \gamma_f h_f + \gamma_m h_m$$

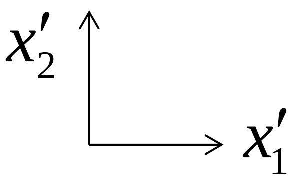
$$\gamma = \frac{W}{h} = \frac{\gamma_f h_f + \gamma_m h_m}{h} = \gamma_f V_f + \gamma_m (1 - V_m)$$

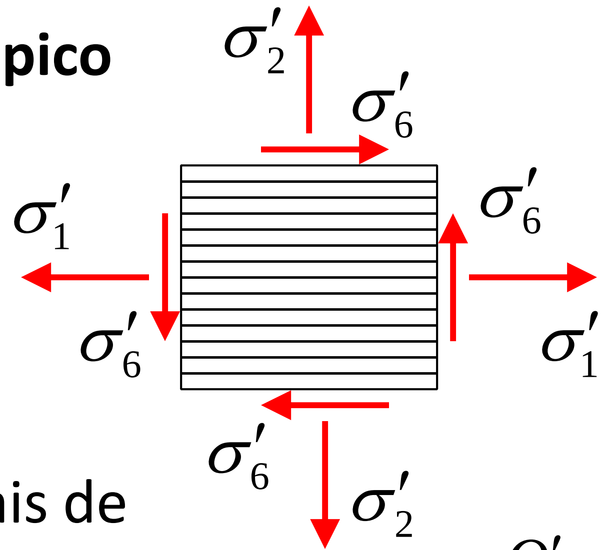
$$\gamma = \frac{\sigma_6}{G_{12}} = \frac{\sigma_6}{G_f} V_f + \frac{\sigma_6}{G_m} (1 - V_m)$$

$$\boxed{G_{12} = \frac{G_f G_m}{G_f (1 - V_f) + G_m V_m}}$$

Estado Plano de Tensão (2D)

Material Ortotrópico





Direções Principais de Simetria do Material

$$\begin{Bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_6' \end{Bmatrix} = \begin{bmatrix} Q_{11}' & Q_{12}' & 0 \\ & Q_{22}' & 0 \\ & & Q_{66}' \end{bmatrix} \begin{Bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_6' \end{Bmatrix}$$

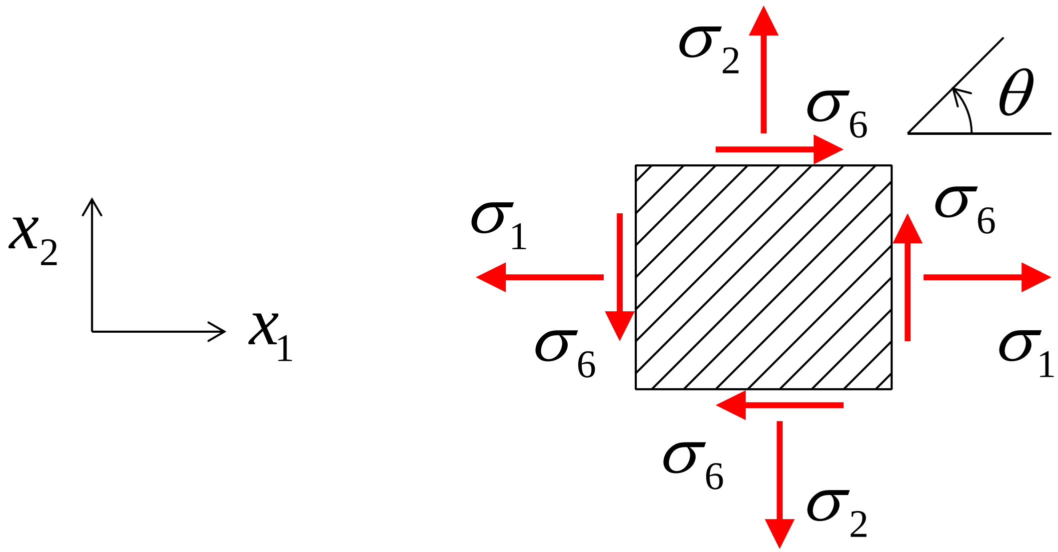
$$Q_{11}' = \frac{E_{11}}{1-\nu_{12}\nu_{21}}$$

$$Q_{22}' = \frac{E_{22}}{1-\nu_{12}\nu_{21}}$$

$$Q_{12}' = Q_{21}' = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}}$$

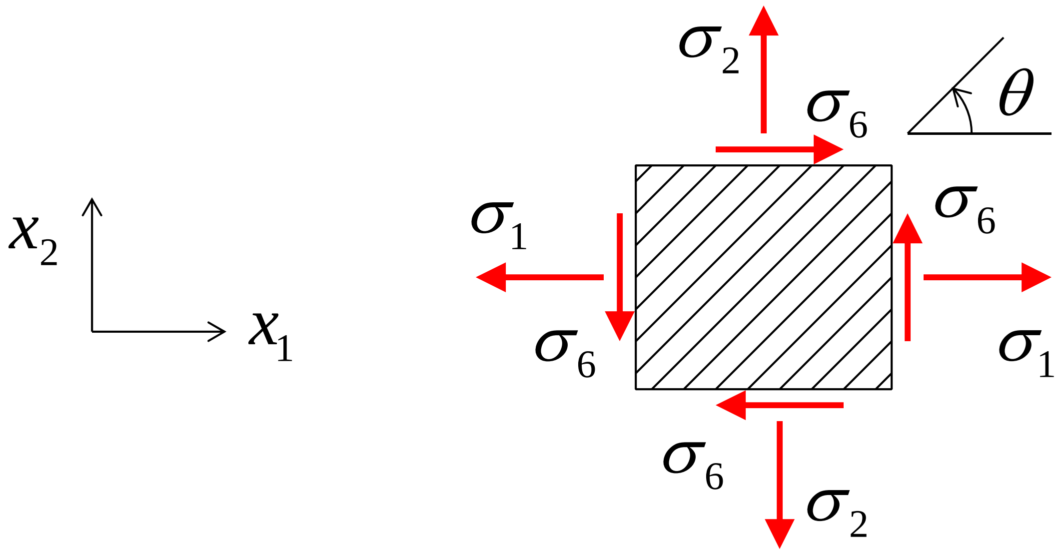
$$Q_{66}' = G_{12}$$

Estado Plano de Tensão (2D)



$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ & Q_{22} & Q_{26} \\ & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

Estado Plano de Tensão (2D)

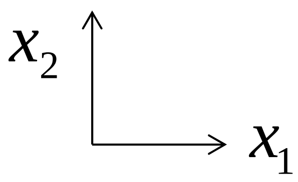


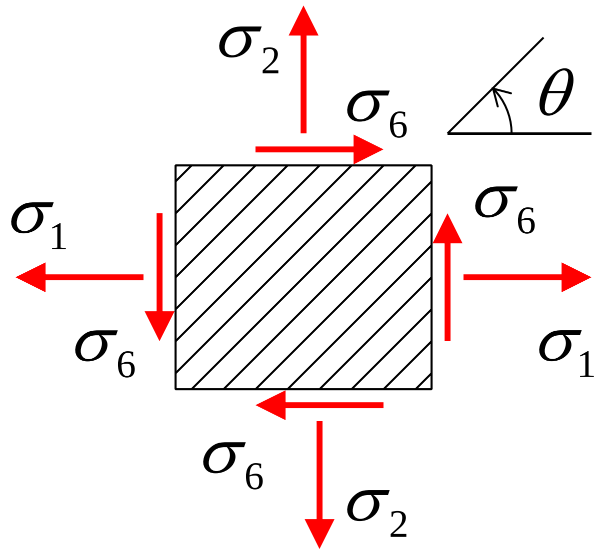
$$[\sigma]=[T][\sigma']$$
$$[\varepsilon']=[T]^T[\varepsilon]$$
$$[\sigma']=[Q'][\varepsilon']$$
$$[\sigma]=[Q][\varepsilon]$$
$$[Q]=[T][Q'][T]^T$$

$$[T]=\begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix}$$

$$m=\cos\theta$$
$$n=\sin\theta$$

Estado Plano de Tensão (2D)





$$Q_{11} = m^4 Q'_{11} + 2m^2 n^2 Q'_{12} + n^4 Q'_{22} + 4m^2 n^2 Q'_{66}$$

$$Q_{22} = n^4 Q'_{11} + 2m^2 n^2 Q'_{12} + m^4 Q'_{22} + 4m^2 n^2 Q'_{66}$$

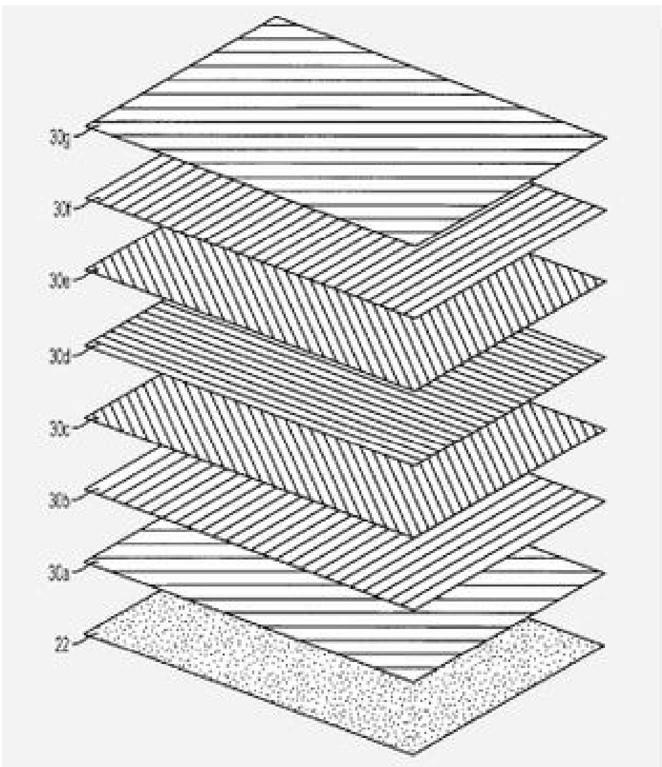
$$Q_{12} = m^2 n^2 Q'_{11} + (m^4 + m^4) Q'_{12} + m^2 n^2 Q'_{22} - 4m^2 n^2 Q'_{66}$$

$$Q_{16} = m^3 n (Q'_{11} - Q'_{12}) - mn^3 (Q'_{12} - Q'_{22}) - 2mn(m^2 - n^2) Q'_{66}$$

$$Q_{26} = mn^3 (Q'_{11} - Q'_{12}) + m^3 n (Q'_{12} - Q'_{22}) + 2mn(m^2 - n^2) Q'_{66}$$

$$Q_{26} = m^2 n^2 (Q'_{11} - Q'_{12}) - m^2 n^2 (Q'_{12} - Q'_{22}) + (m^2 - n^2)^2 Q'_{66}$$

Estado Plano de Tensão (2D)



Propriedades Elásticas no Plano (Lâmina de Material Ortotrópico)

$$E_{11}, E_{22}, G_{12}, \nu_{12}$$

$$\Downarrow$$

$$Q'_{11}, Q'_{22}, Q'_{12}, Q'_{66}$$

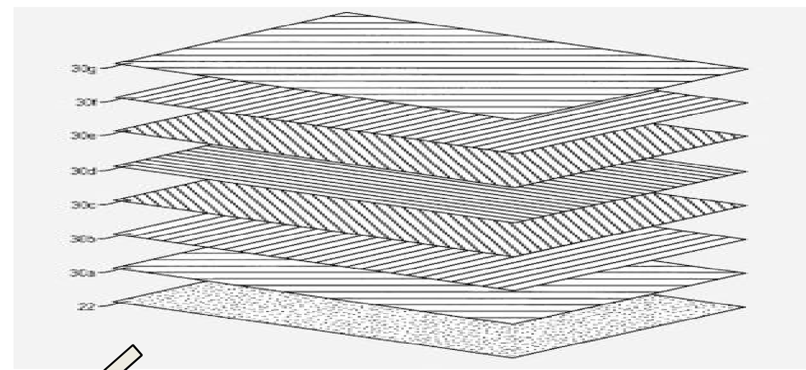
$$\Downarrow$$

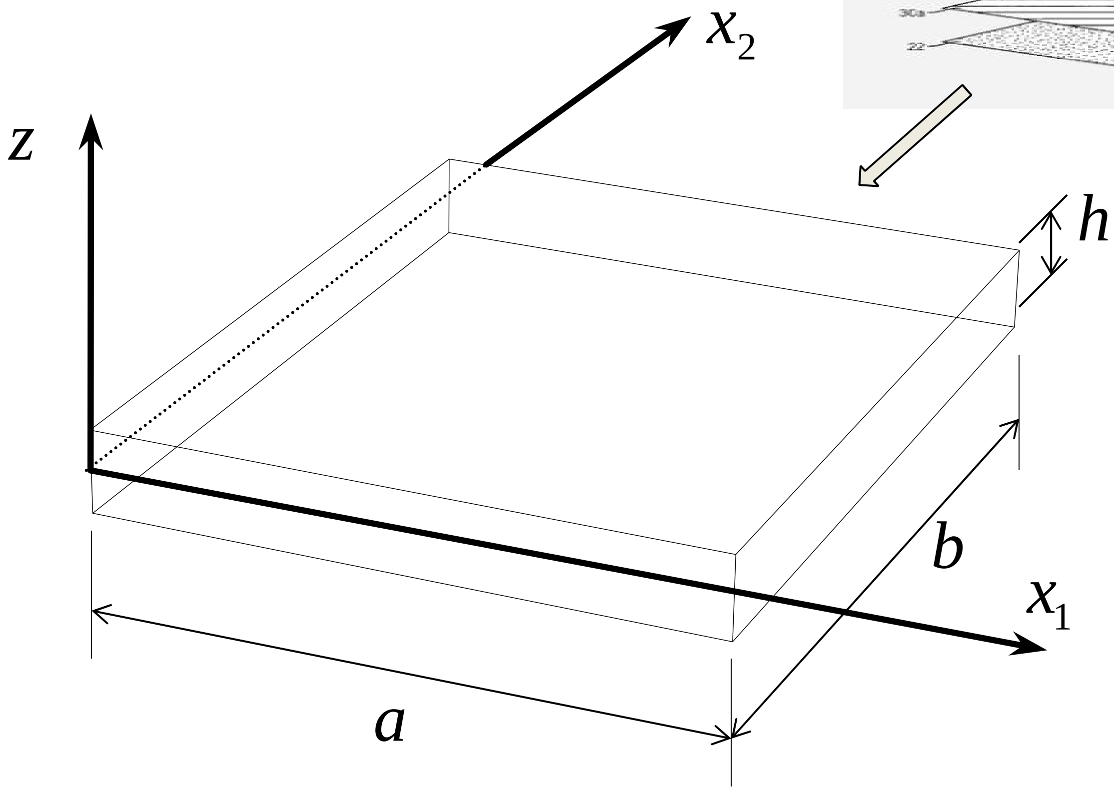
Cada Lâmina $\longrightarrow$ $Q_{11}(\theta), Q_{22}(\theta), Q_{12}(\theta), Q_{16}(\theta), Q_{26}(\theta), Q_{66}(\theta)$

IV. Placas Laminadas

Teoria Clássica de Placas Laminadas

Placas Finas $h << a, b$

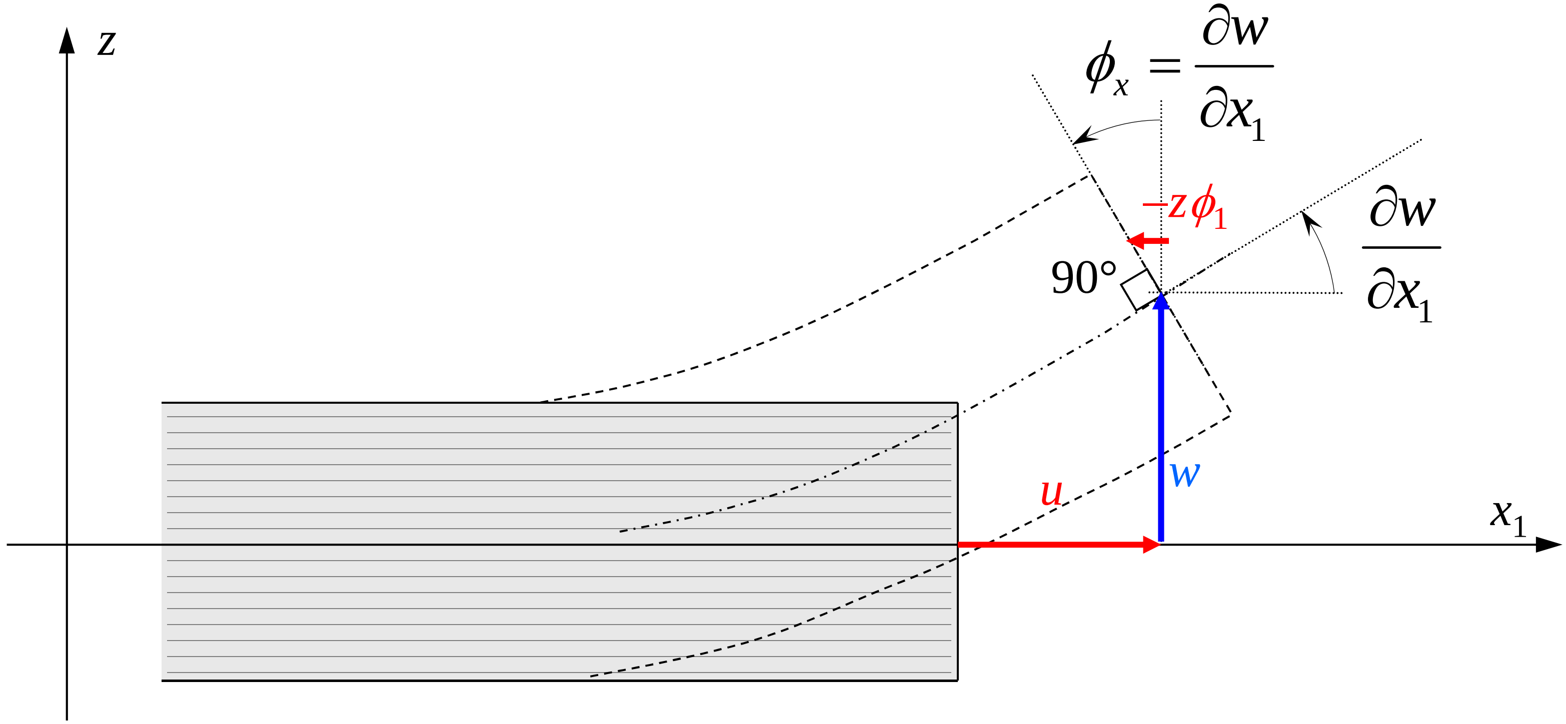




Teoria Clássica de Placas Laminadas

Plano $x_1 z$

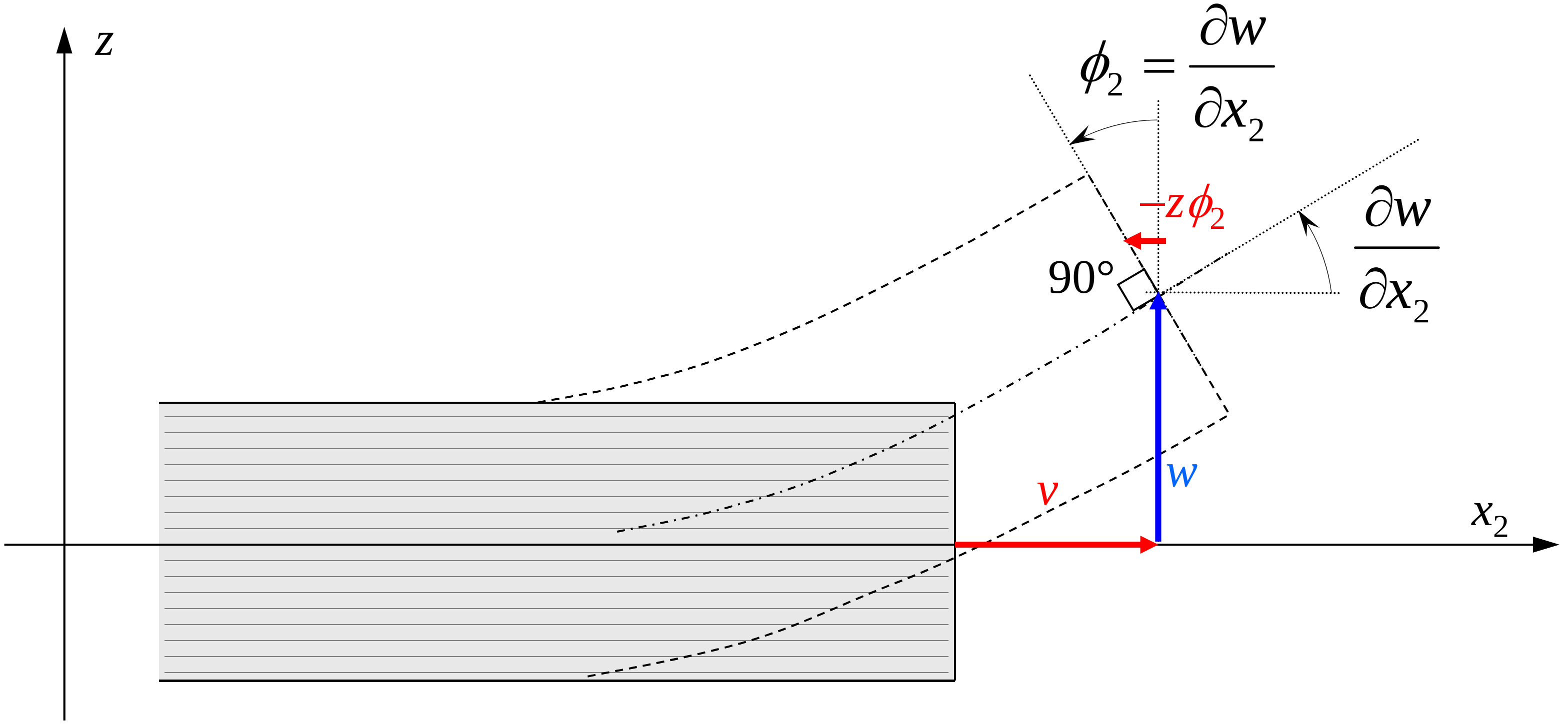
Teoria Clássica
$$\begin{cases} u_1(x_1,x_2,z) = u(x_1,x_2) - z\dfrac{\partial w}{\partial x_1} \\ u_2(x_1,x_2,z) = w(x_1,x_2) \end{cases}$$



Teoria Clássica de Placas Laminadas

Plano $x_2 z$

Teoria Clássica
$$\begin{cases} u_2(x_1, x_2, z) = v(x_1, x_2) - z\dfrac{\partial w}{\partial x_2} \\ u_z(x_1, x_2, z) = w(x_1, x_2) \end{cases}$$



Teoria Clássica de Placas Laminadas

Campo de Deslocamentos

$$u_1(x_1, x_2, z) = u(x_1, x_2) - z\,\partial w/\partial x_1$$
$$u_2(x_1, x_2, z) = v(x_1, x_2) - z\,\partial w/\partial x_2$$
$$u_3(x_1, x_2, z) = w(x_1, x_2)$$

Campo de Deformações

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} = \frac{\partial u}{\partial x_1} - z\,\frac{\partial^2 w}{\partial x_1^2}$$

$$\varepsilon_2 = \frac{\partial u_2}{\partial x_2} = \frac{\partial v}{\partial x_2} - z\,\frac{\partial^2 w}{\partial x_2^2}$$

$$\varepsilon_6 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} - 2z\,\frac{\partial^2 w}{\partial x_1 \partial x_2}$$

Teoria Clássica de Placas Laminadas

Definindo-se

$$\varepsilon_1^0 = \frac{\partial u}{\partial x_1}, \quad \varepsilon_2^0 = \frac{\partial v}{\partial x_2}, \text{ e} \quad \varepsilon_6^0 = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}$$

$$k_1 = -\frac{\partial^2 w}{\partial x_1^2}, \quad k_2 = -\frac{\partial^2 w}{\partial x_2^2}, \text{ e} \quad k_6 = -2\frac{\partial^2 w}{\partial x_1 \partial x_2}$$

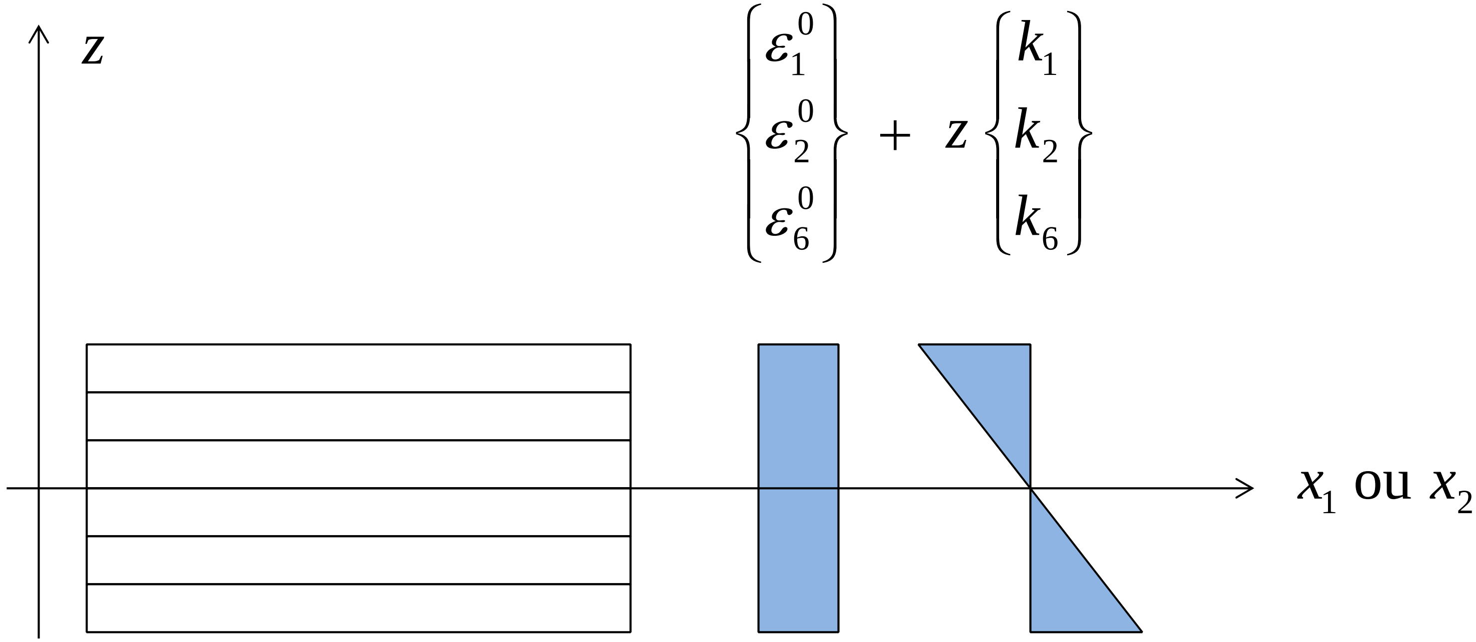
As deformações podem reescritas na forma:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{Bmatrix} k_1 \\ k_2 \\ k_6 \end{Bmatrix}$$

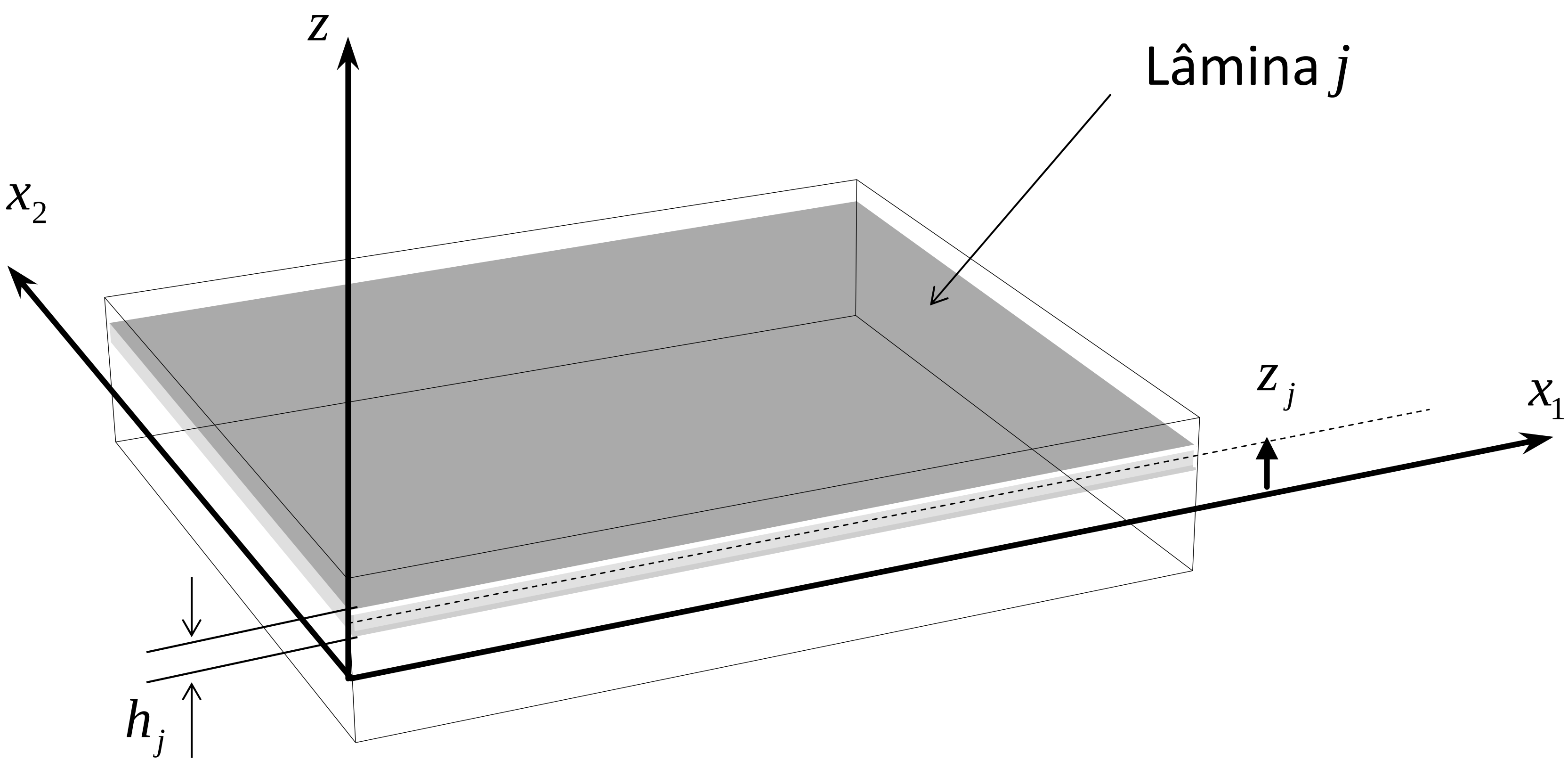
Teoria Clássica de Placas Laminadas

Campo de deformações ao longo da espessura
(linear e contínuo)

$$\begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{Bmatrix} k_1 \\ k_2 \\ k_6 \end{Bmatrix}$$

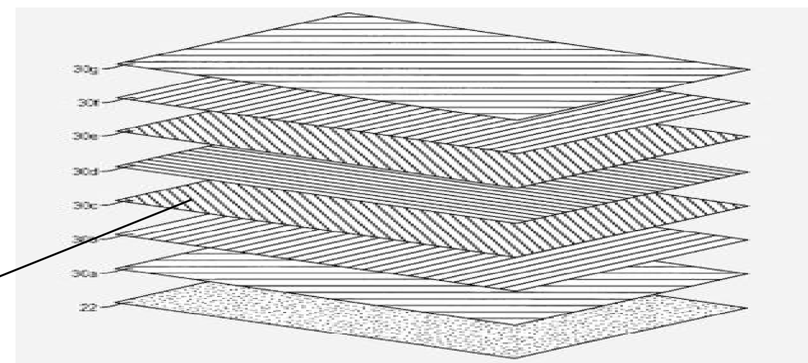


Teoria Clássica de Placas Laminadas



Teoria Clássica de Placas Laminadas

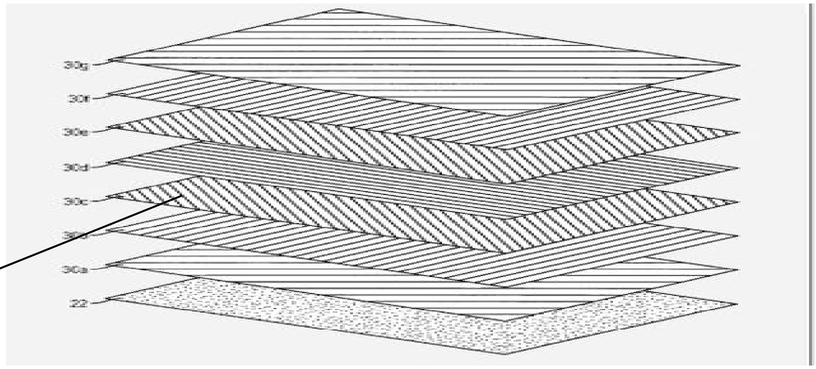
Tensões na Lâmina j ($j = 1,2, \ldots, N$)



$$
\begin{aligned}
\begin{Bmatrix} \sigma_1^{(j)} \\ \sigma_2^{(j)} \\ \sigma_6^{(j)} \end{Bmatrix}
&= \begin{bmatrix} Q_{11}^{(j)} & Q_{12}^{(j)} & Q_{16}^{(j)} \\ Q_{12}^{(j)} & Q_{22}^{(j)} & Q_{26}^{(j)} \\ Q_{16}^{(j)} & Q_{26}^{(j)} & Q_{66}^{(j)} \end{bmatrix}
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \\
&= \begin{bmatrix} Q_{11}^{(j)} & Q_{12}^{(j)} & Q_{16}^{(j)} \\ Q_{12}^{(j)} & Q_{22}^{(j)} & Q_{26}^{(j)} \\ Q_{16}^{(j)} & Q_{26}^{(j)} & Q_{66}^{(j)} \end{bmatrix}
\begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix}
+ z \begin{bmatrix} Q_{11}^{(j)} & Q_{12}^{(j)} & Q_{16}^{(j)} \\ Q_{12}^{(j)} & Q_{22}^{(j)} & Q_{26}^{(j)} \\ Q_{16}^{(j)} & Q_{26}^{(j)} & Q_{66}^{(j)} \end{bmatrix}
\begin{Bmatrix} k_1 \\ k_2 \\ k_6 \end{Bmatrix}
\end{aligned}
$$

Teoria Clássica de Placas Laminadas

Tensões na Lâmina j (j = 1,2, ..., N)



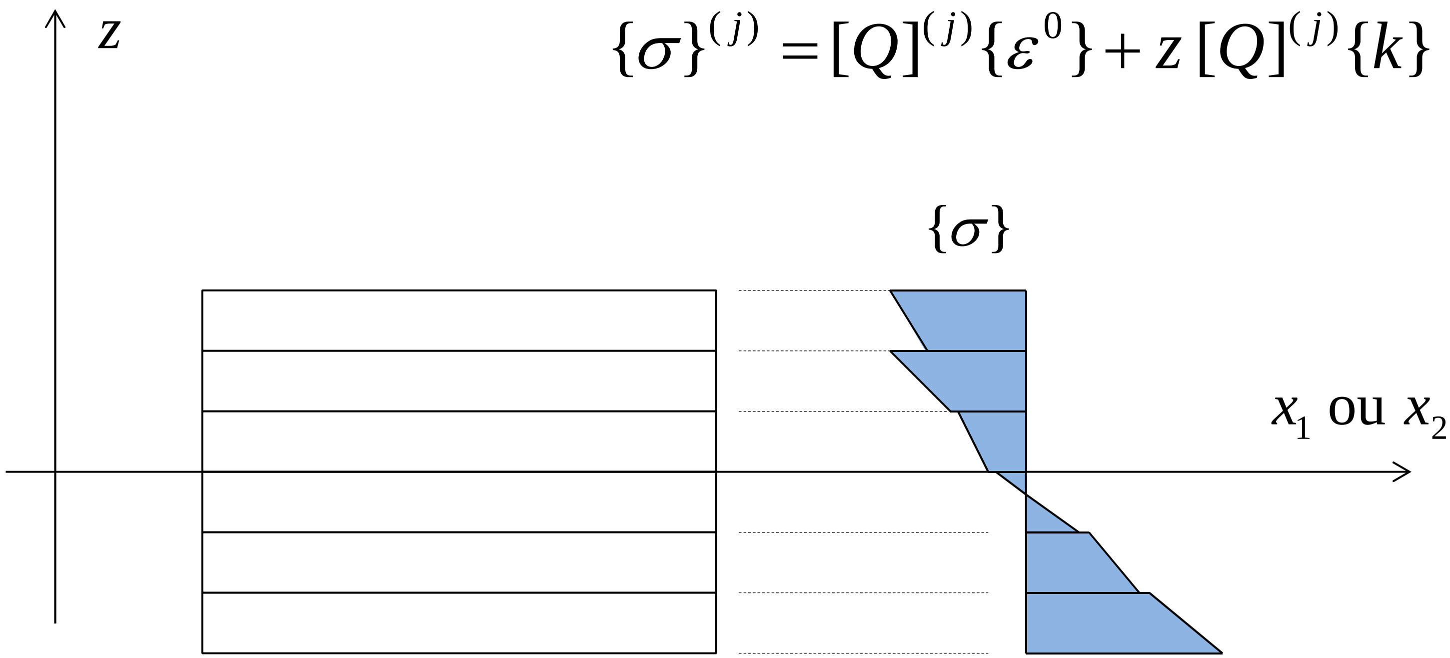
$$\{\sigma\}^{(j)} = [Q]^{(j)}\{\varepsilon^0\} + z\,[Q]^{(j)}\{k\}$$

$$z_j - \frac{h_j}{2} < z < z_j + \frac{h_j}{2}$$

Teoria Clássica de Placas Laminadas

Campo de tensões ao longo da espessura
(linear por partes e descontínuo)

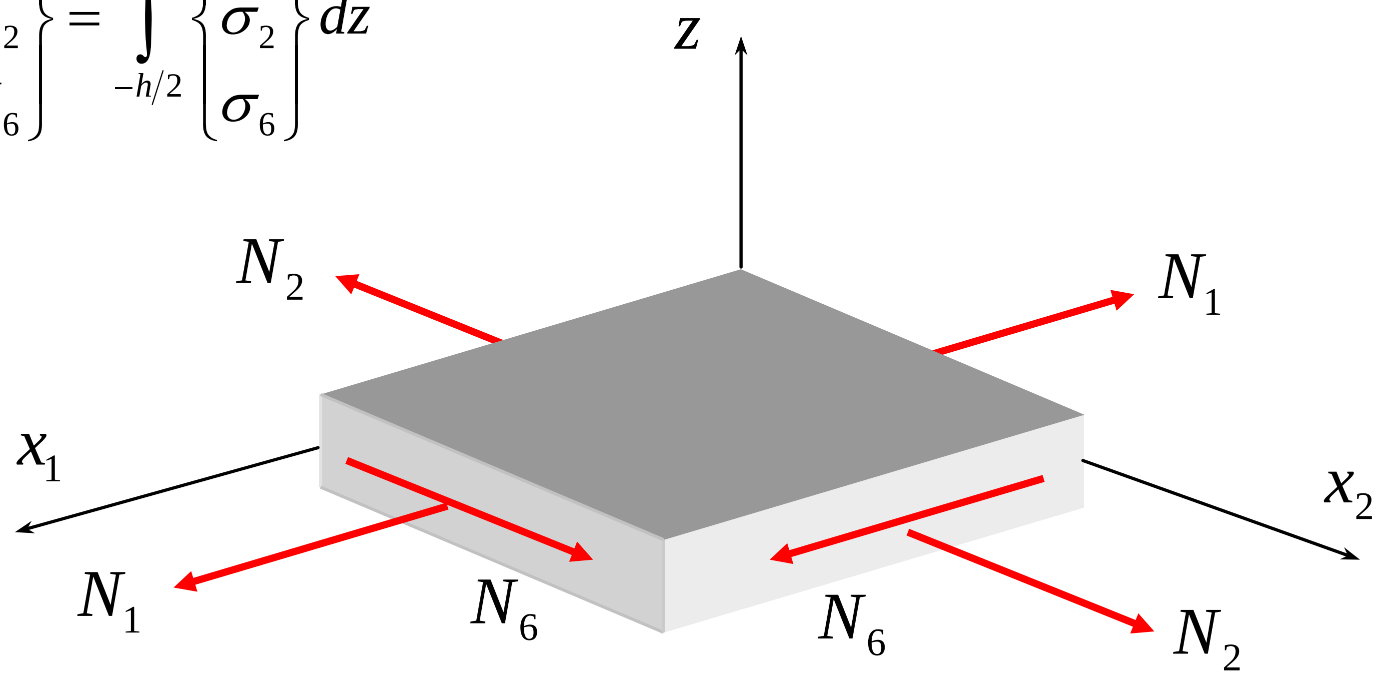
$$\{\sigma\}^{(j)} = [Q]^{(j)}\{\varepsilon^0\} + z\,[Q]^{(j)}\{k\}$$



Teoria Clássica de Placas Laminadas

Esforços Generalizados

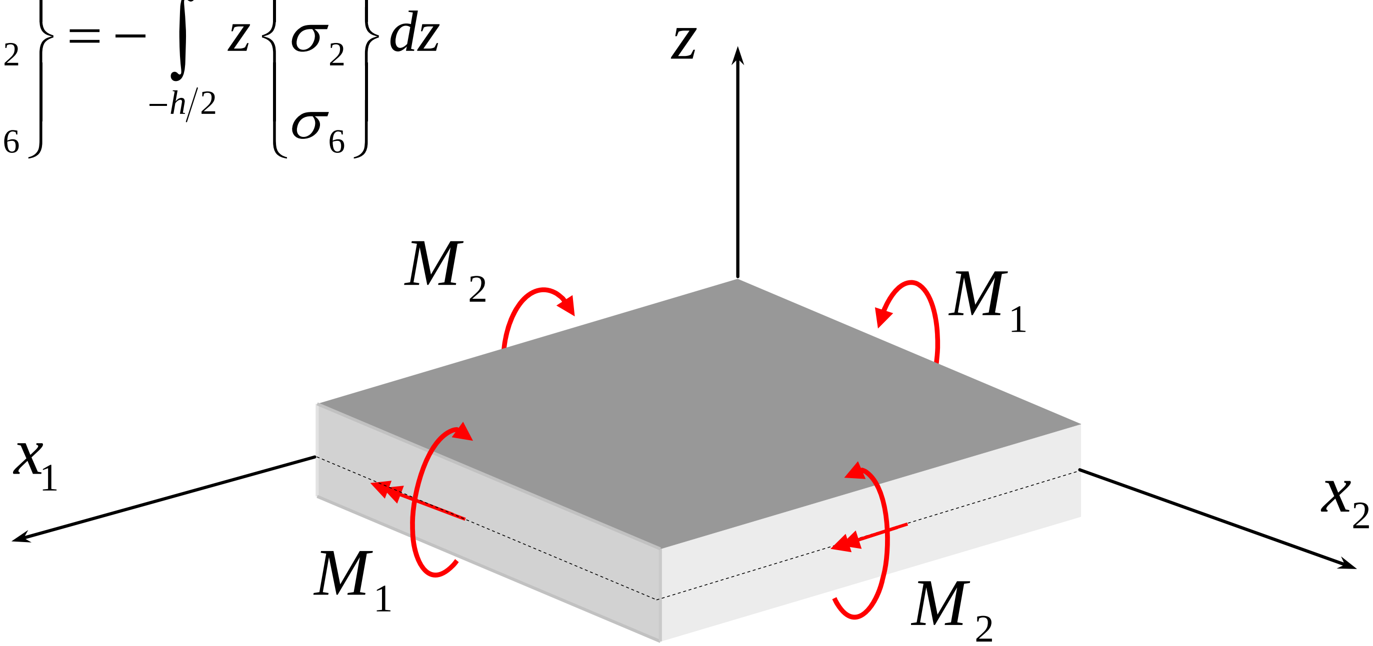
$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz$$



Teoria Clássica de Placas Laminadas

Esforços Generalizados

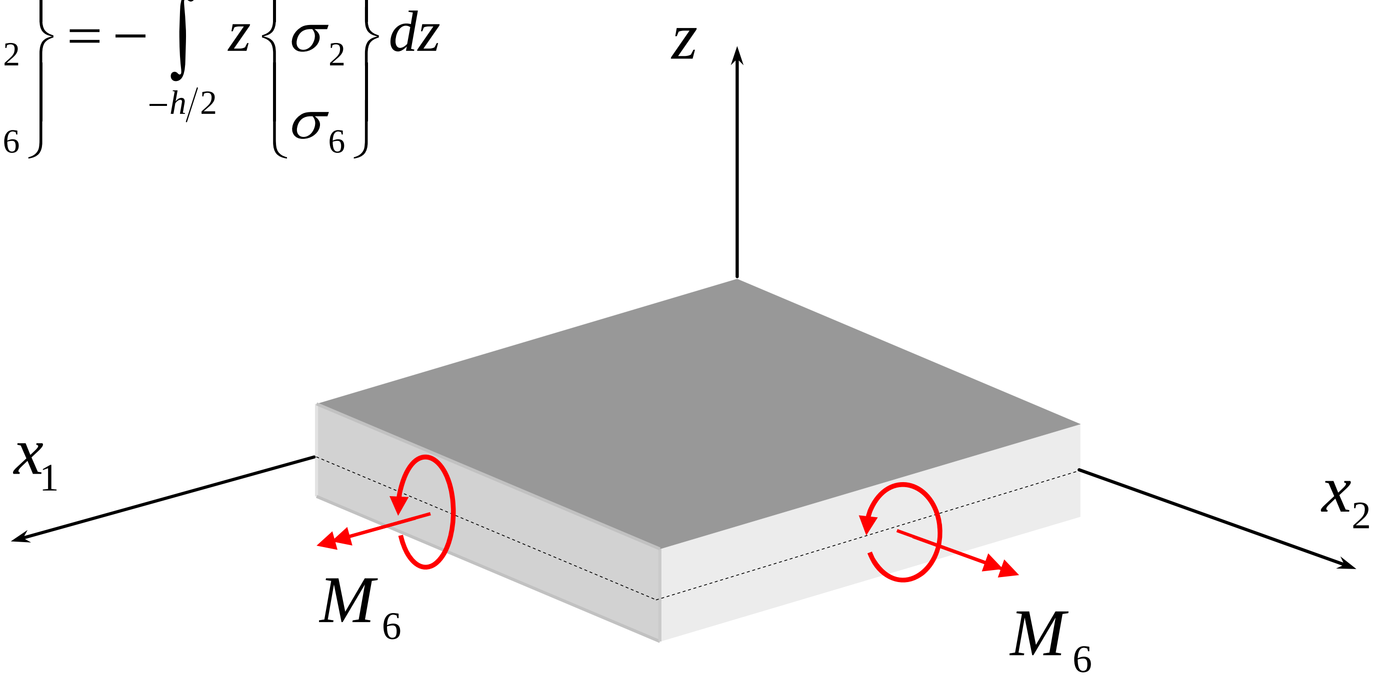
$$\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = -\int_{-h/2}^{h/2} z \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz$$



Teoria Clássica de Placas Laminadas

Esforços Generalizados

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = -\int_{-h/2}^{h/2} z \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz$$



Teoria Clássica de Placas Laminadas

Esforços Generalizados

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz = \sum_{j=1}^{N} \int_{z_j - h_j/2}^{z_j + h_j/2} \begin{Bmatrix} \sigma_1^{(j)} \\ \sigma_2^{(j)} \\ \sigma_6^{(j)} \end{Bmatrix} dz$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = -\int_{-h/2}^{h/2} z \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz = -\sum_{j=1}^{N} \int_{z_j - h_j/2}^{z_j + h_j/2} z \begin{Bmatrix} \sigma_1^{(j)} \\ \sigma_2^{(j)} \\ \sigma_6^{(j)} \end{Bmatrix} dz$$

Teoria Clássica de Placas Laminadas

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{11} & B_{11} & D_{11} & D_{12} & D_{16} \\ B_{11} & B_{11} & B_{11} & D_{12} & D_{22} & D_{26} \\ B_{11} & B_{11} & B_{11} & D_{16} & D_{26} & D_{66} \end{array}\right] \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ k_1 \\ k_2 \\ k_6 \end{Bmatrix}$$

$$A_{pq} = \sum_{j=1}^{N} h_j Q_{pq}^{(j)} \quad (p, q = 1,2,6)$$

$$B_{pq} = 2\sum_{j=1}^{N} z_j h_j Q_{pq}^{(j)}$$

$$D_{pq} = \sum_{j=1}^{N} \left(\frac{12 z_j^2 h_j + h_j^3}{12} \right) Q_{pq}^{(j)}$$